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ABSTRACT

This is part one of a two-part manual for teachers using SMSG high school text materials. A chapter-by-chapter commentary on the text and answers to all the exercises are given. Chapter topics include the idea of the derivative, limits and continuity, differentiation, and applications of the derivative. (MP)

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TEACHER'S COMMENTARY UNIT NO. 68

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CALCULUS

PART I

SE 028 240

SCHOOL MATHEMATICS STUDY GROUP

Calculus

Part 1 Teacher's Commentary

REVISED EDITION

The following is a list of all those who participated in the preparation of this volume:

A. Beck	Olney High School, Philadelphia, Pa.
A. A. Blank	New York University, New York, N.Y.
F. L. Elder	West Hempstead Jr., Sr. High School, N.Y.
C. E. Kerr	Dickinson College, Carlisle, Pa.
M. S. Klamkin	Ford Scientific Laboratory, Dearborn, Mich.
I. I. Kolodner	Carnegie Institute of Technology, Pittsburgh, Pa.
M. D. Kruskal	Princeton University, Princeton, N.J.
C. W. Leeds, III	Berkshire School, Sheffield, Mass.
M. A. Linton, Jr.	William Penn Charter School, Philadelphia, Pa.
H. M. Marston	Douglass College, New Brunswick, N.J.
I. Marx	Purdue University, Lafayette, Ind.
R. Pollack	New York University, New York, N.Y.
T. L. Reynolds	College of William and Mary, Williamsburg, Va.
R. L. Starkey	Cubberley High School, Palo Alto, Calif.
V. Twersky	Sylvania Electronics Defense Labs., Mt. View, Calif.
H. Weitzner	New York University, New York, N.Y.

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Teacher's Commentary

PREFACE

Historically calculus has developed out of the search for methods for solving specific problems. As with all branches of mathematics it is difficult, if not impossible, to fix upon a particular period in history as the moment of discovery or invention. It is especially hard in the case of calculus since the problems that stimulated the creation of the subject had existed for thousands of years. Initially the techniques were viewed pragmatically; if they produced reasonable answers, they were considered sound. Subsequently, others analyzed the procedures in detail and pointed out inadequacies. Nevertheless, it was through such analysis that the full power of the pioneer methods was realized.

The SMSG Calculus does not attempt to parallel the historical development but it is rooted in the search for solutions to problems. Problems are used to reveal the important issues. These issues are then formulated in terms of (sometimes incomplete) mathematical models. The models, in turn, suggest considerations that motivate the study of the theoretical structure of the course. Then these ideas must be expressed precisely so that we may reason about them logically. Finally, the course returns again to problems and applies the theory which has been developed.

Teacher's Commentary

FOREWORD

The calculus unifies geometric and algebraic strands of earlier courses and offers the most general preparation for further study of mathematics and its applications. It is the natural capstone of the secondary school mathematics program.

Calculus is in transition from the colleges to the high schools. Now that increasingly many superior students complete the SMSG or a comparable curriculum by the end of the eleventh grade, the trend toward a calculus course offered in the twelfth grade is accelerating. Students taking such a course usually hope for advanced standing in college. On the other hand, colleges are intensifying demands upon these students. Higher institutions are reluctant to allow advanced standing for prior work in the calculus on the ground that it may not provide an adequate conceptual basis for further work in mathematics.

The immediate purpose of this text is to bridge the transition from high school to college by supplying a one year calculus course which provides an adequate conceptual basis for advanced standing in the programs of leading institutions and, at the same time, is designed to meet the needs of high school students and teachers. The course has not been bound to the syllabus of the Advanced Placement Program of the College Entrance Examination Board since that syllabus can only reflect the curriculum of the moment. Even so, all the significant calculus topics are covered in depth; a student who masters our suggested minimal course should do well on the calculus questions of past Advanced Placement Examinations.

We expect that the student who takes calculus in high school is sufficiently prepared to undertake a course which encourages a mature attitude concerning analysis.

The authors hope to involve the student in the exciting enterprise of exploring a living subject. Throughout the text they address the student as a mature person, curious and vitally interested in relating to other disciplines the immediate knowledge to be acquired. His desire to order the world in a rational system needs to be satisfied: he must be concerned with the calculus as it was conceived and continues to grow. For that reason any traditional material of doubtful current utility is pruned or even omitted altogether in this book. On the other hand, novelty for its own sake is avoided; although many matters in this text are unconventional, none are included for that reason.

The course is a continuation of the SMSG series of texts and may be used as a sequel to Intermediate Mathematics. Appendices 1 and 2 include a recapitulation of essential ideas to strengthen real number and function concepts pertinent to the calculus.

We appeal to the student's intuition in the development of the text. The carefully selected range of problems together with the analytical discussion of questions, simple and profound, in the appendices invite the most discriminating reader to explore and pursue according to his individual abilities, stamina, and interests.

No absolute time scale is suggested because the course does not correspond exactly to traditional syllabi. The total coverage in a year course will vary greatly. A rich treatment pursuing depth rather than breadth might use the full year to cover the fundamental calculus of Chapters 1-8 together with some of the related appendices. Preliminary experience indicates that Chapters 1-11, which correspond approximately to the scope of the traditional one year course, can and will be covered by a substantial fraction of classes using the texts and many will want to push on further into the later chapters.

We should like to know how the approach of this text works out in the classroom, to know your reactions and those of your students. We invite you to tell us of our errors of omission and commission.

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Teacher's Commentary

Chapter 1

INTRODUCTION

Chapter 1 sets the tone for the course. The initial investigation is exploratory, tentative, and incomplete. The exploratory phase is the time of discovery, the time of development of intuitions and perceptions. Later we shall frame precise definitions and rigorous proofs. This is part of the general plan of the text. As the course progresses, the span between the initial exploratory phase and the ultimate phase of rigorous proof is shortened.

The chapter must not be overdone (see Suggested Time Schedule). The air will be full of questions which cannot be laid to rest before the course has been completed. The student is not expected to understand all of the calculus after reading the introduction; the idea is only to excite his interest and curiosity and stimulate his awareness of the fact that he is embarking on a new and interesting subject.

TC1-1. Best Value Problems. The Derivative.

In the "parcel post" example in the text which begins on page 2 we assume that a solution to the problem exists. Later we shall establish the existence of a solution.

We give here another procedure for showing that a square cross-section is best for a given girth. If we denote the length, width, and height of the carton by ℓ , w , and h , respectively, then the girth is $2w + 2h$ and our problem is to maximize the volume

$$V = \ell wh$$

where ℓ , w , h satisfy the conditions

$$\ell + 2w + 2h = 72 ,$$

$$\ell \geq w \geq 0 , \quad \ell \geq h \geq 0 .$$

We reduce this problem to one of maximizing a function of a single variable by using a problem-solving technique known as "relaxation." Let us first assume that we know the correct value of ℓ (for a maximum, say $\overline{\ell}$). Our problem would then be to maximize wh, where w and h satisfy

$$2w + 2h = 72 - \overline{\ell} ,$$

$$w > 0 , \quad h > 0 .$$

It follows that $w = h$. (See footnote, p.   of this text or SMSG, Intermediate Mathematics, p. 214, Ex. 4, p. 216, Ex. 4).

Exercises 1-1

In Section 1-1 we emphasize the problem of locating the maximum, but in general, the most difficult part of maximum and minimum problems consists of finding the function to maximize or minimize. Since no general rule can be given for doing this, the student may find some of the problems in Exercise 1-1 quite challenging.

Essentially, the question involves the ability to translate an English statement to a mathematical equivalent (SMSG, First Course in Algebra, Chapter [illegible], familiarity with basic mensuration formulas of geometry, and a sound understanding of the concept of a function and related ideas (Appendix 2).

These problems are included at this time as a preliminary background for the solution of the same and similar problems in Chapter 5. It is not necessary (and certainly not suggested) that the student complete each problem in Exercise 1-1 before going on to the next section. Spiral assignments are recommended; mathematical maturity and an appreciation of functional relationships are factors in the student's success with problems of this sort. If used judiciously these problems will motivate the course; if used indiscriminately they may cause indigestion.

In Exercises 1-1 the student is asked to write an equation describing the function and is not required to solve the given problem. If he is curious about the solution he should be encouraged to use the procedures of Section 1-1 to approximate the answer. Although he runs into difficulties he will be led to appreciate the limitations of algebra and the need for more powerful methods.

Problems 12 and 13 are included at this time to give the student preliminary experience with graphs of polynomial functions. Hasty sketching, without examination of functional values, may lead to incorrect conclusions.

Solutions Exercises 1-1

1. Express the area of a semicircle as a function of its perimeter.

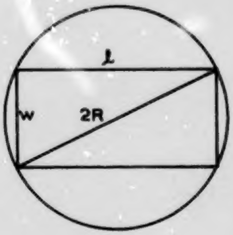
For a semicircle $A = \frac{\pi r^2}{2}$, and $P = r(\pi + 2)$.

$$A = f(P) = \frac{\pi \cdot P^2}{2(\pi + 2)^2} .$$

2. A rectangle is inscribed in a circle of radius R . Express the area of the rectangle as a function of the length of one side.

$$4R^2 = \ell^2 + w^2 \qquad \ell = \sqrt{4R^2 - w^2}$$

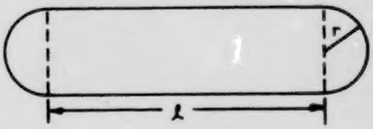
$$A = \ell w = (\sqrt{4R^2 - w^2}) \cdot w = f(w) .$$



3. A tank has the form of a right circular cylinder capped by hemispherical ends. If its total volume is V , express the length of the cylinder as a function of its radius.

$$V = \pi r^2 \ell + \frac{4}{3} \pi r^3$$

$$\ell = \frac{V}{\pi r^2} - \frac{4r}{3} = f(r) .$$



4. Determine the edge of a regular tetrahedron inscribed in a sphere as a function of the radius R .

Let C be the center of the sphere and A , B , G , D the vertices of the inscribed regular tetrahedron of edge e . Since

$$R = BC = GC = DC$$

then C is on the line perpendicular to plane BGD at the centroid F of $\triangle$ BGD . From the figure it also follows that

$$e = AB = AG = AD$$

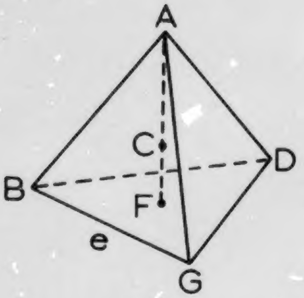
and A is on the line perpendicular to plane BGD at the centroid F of $\triangle$ BGD . By definition of the centroid of an equilateral triangle,

$$BF = \frac{2}{3}\frac{\sqrt{3}}{2}e = \frac{e}{\sqrt{3}} .$$

Then $\quad AF = \sqrt{e^2 - BF^2} = \frac{\sqrt{6}}{3} e ,$

but $\quad AF = R + CF .$

Therefore, $\quad R + CF = \frac{\sqrt{6}}{3} e .$



From the Pythagorean relation

$$CF^2 + BF^2 = R^2$$

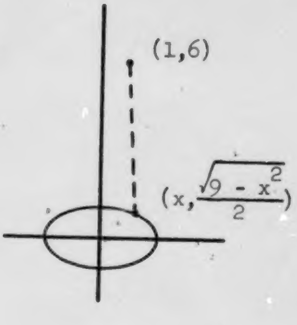
whence

$$CF = \frac{1}{3}\sqrt{9R^2 - 3e^2}$$

so

$$e = f(R) = \frac{2}{3}\sqrt{6}\, R .$$

5. What point on the ellipse $x^2 + 4y^2 = 9$ is nearest the point $(1,6)$? Let D be the distance from the point $(1,6)$ to any point $\left(x, \frac{\sqrt{9 - x^2}}{2}\right)$ of the ellipse



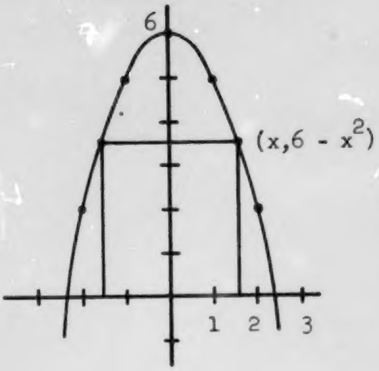
$$D^2 = (x - 1)^2 + (\tfrac{1}{2}\sqrt{9 - x^2} - 6)^2$$

$$= \frac{157}{4} - 6\sqrt{9 - x^2} - 2x + \frac{3x^2}{4} = f(x).$$

Since $D > 0$ we can minimize D^2 and thus minimize D .

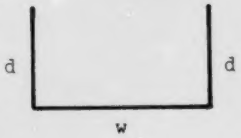
6. A rectangle has two of its vertices on the x-axis and the other two above the axis on the parabola $y = 6 - x^2$. What are the dimensions of such a rectangle if its area is to be a maximum?

$$\begin{aligned} A &= \ell w \\ &= 2xy \\ &= 2x(6 - x^2) = f(x)\ . \end{aligned}$$



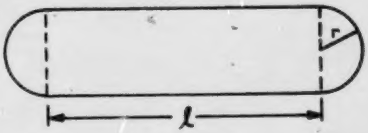
7. A rectangular sheet of galvanized metal is bent to form the sides and bottom of a trough so that the cross section has this shape: ⊔ . If the metal is 14 inches wide, how deep must the trough be to carry the most water?

Let d be the depth and w the width. Then $2d + w = 14$. For the cross-sectional area A , we have $A = dw$. The function to be maximized is $A = f(d) = 2d(7 - d)$.



8. The tank described in exercise 3 above is to be made in the most economical way. If the material for the hemispherical ends costs twice as much per square foot as that for the cylindrical part, find the most economical dimensions. (That is, find "cost" as a function of either the radius or the length.)

Surface area of hemisphere = $2\pi r^2$. Surface area of two ends is $4\pi r^2$. Surface area of cylindrical part is $2\pi r\ell$. Let cost be 1 unit per square foot for cylindrical part, and 2 units per square foot for ends.



$$C = 2(4\pi r^2) + 2\pi r\ell\ .$$

From Problem 3

$$\ell = \frac{V}{\pi r^2} - \frac{4r}{3} = \frac{V - \frac{4}{3}\pi r^3}{\pi r^2}$$

(presumably, $\ell > 0$, so $\frac{4}{3}\pi r^3 < V$) .

So,

$$C = f(r) = 8\pi r^2 + 2\pi r \left[\frac{V - \frac{4}{3}\pi r^3}{\pi r^2}\right]$$

$$= 2\left[\frac{8}{3}\pi r^2 + \frac{V}{r}\right]$$

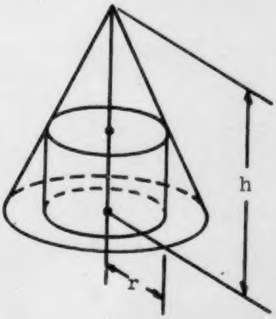
is the function to be minimized, where the domain of f is $0 < r < (\frac{3V}{4\pi})^{\frac{1}{3}}$.

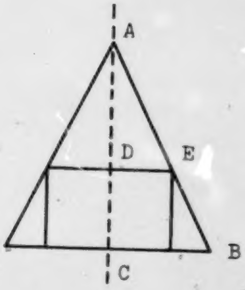
9. Find the right circular cylinder of greatest volume that can be inscribed in a right circular cone of radius r and height h.

Let h be the height of the given cone, and r the radius of its circular base. Then in terms of the cross-sectional sketch, $AC = h$ and $CB = r$. If the height of the inscribed cylinder is y, and the radius of its circular base is x, then $DC = y$ and $DE = x$. Since triangles ABC and AED are similar, $\frac{DE}{AD} = \frac{CB}{AC}$, or $\frac{x}{h - y} = \frac{r}{h}$, where $AD = h - y$, and $y = h - \frac{hx}{r}$. The volume V of the cylinder is given by

$$V = \pi x^2 y = \pi h x^2 - \frac{\pi h x^3}{r} = f(x),$$

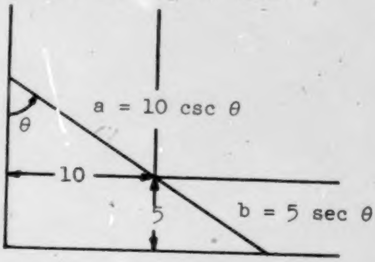
the polynomial function f which we can maximize.





10. Find the length of the longest rod which can be carried horizontally around a corner from a corridor 10 ft. wide into one 5 ft. wide.

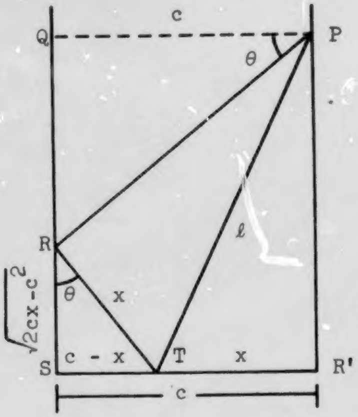
The class might be shown this drawing. Then after obtaining the "length" as a function of θ, the question should be raised about whether we want the minimum or maximum value of this function. (It is the minimum.)



Let L be the length of the rod, and θ the angle indicated,

$$L = f(\theta) = 5(2 \csc\theta + \sec\theta).$$

11. The lower right-hand corner of a page is folded over so as to reach the left edge in such a way that one endpoint of the crease is on the right edge of the page and the other endpoint is on the bottom of the page as in the figure.



Let ℓ be the length of the crease. Since $\triangle PQR \sim \triangle RST$, we have $\frac{PR}{PQ} = \frac{RT}{RS}$ and $PR = \frac{cx}{\sqrt{2cx - c^2}}$.

From right $\triangle PRT$,

$$\ell^2 = \left(\frac{cx}{\sqrt{2cx - c^2}}\right)^2 + x^2$$

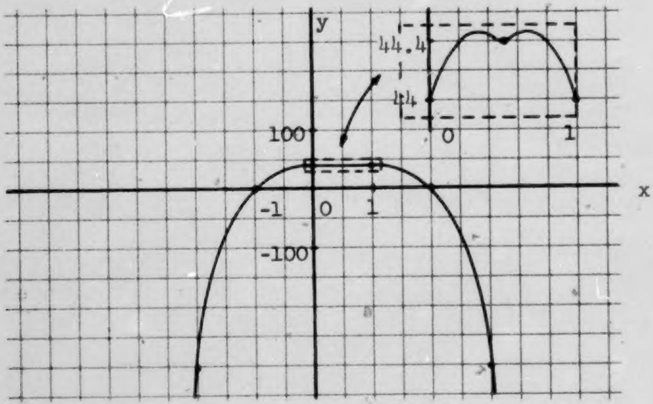
$$= \frac{2cx^3}{2cx - c^2}.$$

We can set $\ell^2 = f(x) = \frac{2x^3}{2x - c}$.

Since $\ell > 0$ we can minimize ℓ^2 and thus minimize ℓ.

12. For $f(x) = 44 + 4x - 13x^2 + 18x^3 - 9x^4$ we give certain of the functional values and a graph on which the points of the table are shown. Sketch a graph through the points. Where do you think the maximum value of f is?

x	-2	-1	0	1	2	3
f(x)	-304	0	44	44	0	-304



The maximum value of $f(x)$ is at $f(\frac{1}{3}) = f(\frac{2}{3}) = 44\,\frac{4}{9}$.

$$f(\tfrac{1}{2}) = 44\,\tfrac{7}{16} < 44\,\tfrac{4}{9}.$$

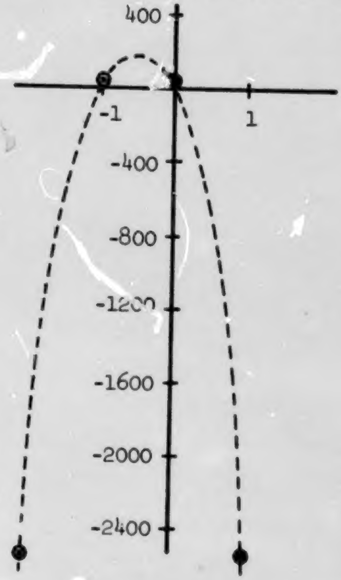
13. Approximate the maximum value of the function

$$f(x) = 39 - 640\,x^2 - 1280\,x^3 - 640\,x^4\,.$$

In Ex. 12, the maximum value of x could be estimated reasonably well from the crude plotting of points since coincidentally the graph of $y = f(x)$ is smooth and relatively flat in the interval $[0,1]$ in which the maximum occurs. Without any further information, just the crude plotting of points of a graph is often misleading. Here is an illustration of this. If we plot points as before we obtain:

x	-2	-1	0	1
$f(x)$	-2520	39	39	-2520

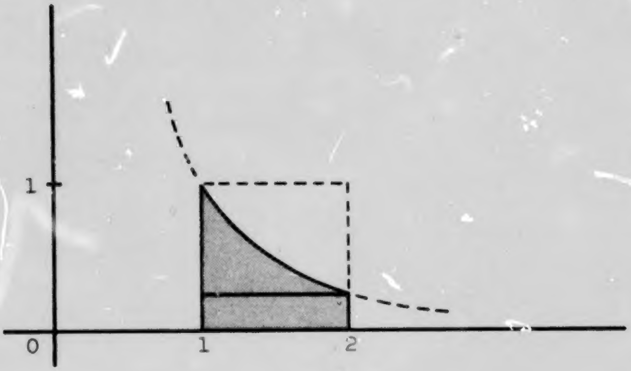
and sketching in a smooth curve symmetric with $x = -\frac{1}{2}$ could lead to an approximation of the maximum value which would be wildly wrong. Actually $f(x)$ has a maximum at both $f = -1$ and 0 and a relative minimum value of -39 at $x = -\frac{1}{2}$. Subsequently, we will return to this problem after sufficient background has been developed in Chapter 2.



Solutions Exercises 1-2

1. Sketch the graph of $f : \longrightarrow \frac{1}{x^2}$ from $x = 1$ to $x = 2$.

x	1	$\frac{6}{5}$	$\frac{7}{5}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{9}{5}$	2
$f(x)$	1	$\frac{25}{36}$	$\frac{25}{49}$	$\frac{4}{9}$	$\frac{25}{64}$	$\frac{25}{81}$	$\frac{1}{4}$



(a) Estimate the integral A of f from 1 to 2 using any method that you wish.

$$\frac{1}{4} \leq A \leq 1 .$$

(b) If you use the method of Section 1-2 and approximate A by rectangular areas, what are upper and lower estimates for two subintervals?

The student may choose to subdivide the interval at any point x_0 such that $1 < x_0 < 2$. We choose $x_0 = 1.5$. Then

$$\frac{4}{9}\left(\frac{3}{2} - 1\right) + \frac{1}{4}\left(2 - \frac{3}{2}\right) \leq A \leq 1\left(\frac{3}{2} - 1\right) + \frac{4}{9}\left(2 - \frac{3}{2}\right)$$

or

$$\frac{25}{72} \leq A \leq \frac{13}{18} .$$

What is the maximum error E_2 for this subdivision? $E_2 = \frac{3}{8}$.

(c) Using the same procedure, subdivide the base into n equal parts and find the maximum error E_n for this subdivision.

If we denote the successive endpoints $x_0, x_1, \ldots, x_n$ where $x_0 = 1 = \frac{n}{n}$, $x_1 = 1 + \frac{1}{n} = \frac{n+1}{n}$, $x_2 = 1 + \frac{2}{n} = \frac{n+2}{n}$, ... , $x_n = 1 + \frac{n}{n} = \frac{2n}{n}$, then

$$f(x_1)(x_1 - x_0) + f(x_2)(x_2 - x_1) + \dots + f(x_n)(x_n - x_{n-1}) \le A$$

and

$$A \le f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + \dots + f(x_{n-1})(x_n - x_{n-1})$$

or

$$\frac{1}{n}(f(x_1) + f(x_2) + \dots + f(x_n)) \le A \le \frac{1}{n}(f(x_0) + f(x_1) + \dots + f(x_{n-1})) .$$

$$\text{Maximum error} = E_n = \frac{1}{n}[f(x_0) - f(x_n)] = \frac{3}{4n} .$$

(d) How can the maximum error E_n be brought below the error tolerance 0.03?

$$\frac{3}{4n} < .03 \quad \text{for } n \text{ a natural number}$$

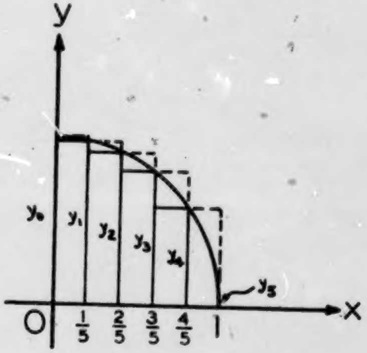
$$n > 25 .$$

(e) How can the maximum error E_n be brought below any given error tolerance?

$$\frac{3}{4n} < T \quad \text{where } T \text{ is the tolerance.}$$

Choose n such that $nT > \frac{3}{4}$.

2. A circle of unit radius has an area of π square units. Consequently, we can approximate π by using the same method as in Ex. 1. Here we shall use a quarter of a circle and obtain an approximation for $\frac{\pi}{4}$. If we use five equal subdivisions, the rectangles enclosed in the desired region will have heights of y_1 , y_2 , y_3 , y_4 , and y_5 , while the rectangles containing the desired region will have altitudes of y_0, y_1 , y_2 , y_3 , y_4 , where y_i is the ordinate of the quarter-circle $y = \sqrt{1 - x^2}$ at $x = \frac{i}{5}$, $i = 0, 1, \dots, 5$. The base of each rectangle is $\frac{1}{5}$; consequently,



$$\frac{1}{5}(y_1 + y_2 + y_3 + y_4 + y_5) < \frac{\pi}{4} < \frac{1}{5}(y_0 + y_1 + y_2 + y_3 + y_4) .$$

The difference between these larger (ℓ) and smaller (s) estimates is

$$\frac{y_0 - y_5}{5} \text{ or } \frac{1}{5} .$$

Then the average of the larger and smaller estimates can only differ from $\frac{\pi}{4}$ by less than half this difference; that is,

$$\left|\frac{\pi}{4} - \frac{\ell + s}{2}\right| < \frac{1}{10} .$$

How many equal subdivisions should one take to obtain an approximation of π correct to within $\pm .01\%$ of π ? If we use n equal subdivisions, then $\ell - s = \frac{1}{n}$ and

$$\left|\frac{\pi}{4} - \frac{\ell + s}{2}\right| < \frac{1}{2n} \quad \text{or}$$

$$|\pi - 2(\ell + s)| < \frac{2}{n} .$$

We want $\frac{2}{n} \leq .0001\,\pi$

and so $n \geq \frac{2}{.0001\,\pi}$.

Assuming we know that $\pi > 3$, a suitable value of n is then $\frac{2}{.0003}$ or 6,667.

This method of approximating π is not particularly efficient. Subsequently, we will give a much more efficient scheme.

Solutions Exercises 1-3

The following is a quotation from G. Polya: "If you cannot solve the proposed problem, try to solve first some related problem."[*] In the problem of Section 1-1 concerning the shipping of books a "related problem" is considered when we switch our unfocused interest in the problem of finding the dimensions of a box with maximum capacity to the specific related problem of determining properties of the function f given by

$$f(x) = V = x^2(72 - 4x) .$$

A final related problem is considered when looking for the point of the graph of this function at which the slope is zero.

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[*] Polya, G., How To Solve It, 2nd ed., New York: Doubleday, 1956; p. 10.

What are the related problems in (a) Ex. 1-1, number 5?
(b) Ex. 1-1, number 7?
(c) Ex. 1-1, number 8?
(d) Ex. 1-1, number 10?
(e) Ex. 1-2, number 1?

These are discussion problems for which there are numerous possible answers. Discuss as many of these as time permits. Ask students to indicate which step in their judgment was the most crucial in finding an answer and which ideas were new to them.

Teacher's Commentary

Chapter 2

THE IDEA OF DERIVATIVE

One of the two basic ideas of the elementary calculus is the derivative. It is easy to appreciate this idea intuitively and know why it is useful before formulating it precisely.

This discussion begins at the heuristic level with the geometric interpretation of the derivative of a function at a point. The slope of a curve at a point is developed as a limit of slopes of all chords passing through a given point on the graph of the function. The student is led to examine the slope of a chord joining the point (a,b) of the graph to any other point (x,y) on the graph. If $|x - a|$ is sufficiently close to zero, this slope is accepted as an approximation to the slope of the graph at (a,b). The question of what is meant by sufficiently close to zero is the motivation for the definition of limit as developed in Chapter 3.

The point that is being stressed is that $r(x)$ represents the slope of a chord and has no meaning if $x = a$. The fact that some students may wish to bypass the analysis and arrive at what appears to be the same result by mere algebraic manipulation (cancellation and substitution) indicates a need to look at the general development a little more carefully.

The general quadratic function is used as a model for Section 2-3 under the assumption that the student is familiar with an elementary analysis of this function. If this is not so, you may refer him to Chapter 4 of SMSG Intermediate Mathematics.

Velocity is introduced (Section 2-4) as a second interpretation (application) of the derivative. At this time the primary aim is to have the student think of instantaneous velocity as a limit which takes on the same form as the slope of the graph of a function at a point.

The concept of a tangent line to a curve has been deferred until Section 5-7.

The phrase "as x approaches a" is to be understood in the sense "as x approximates a." We never permit x to assume the value a in the approximation process.

The word "approximation" is essential to an attitude about limits which the student is encouraged to adopt. It is established that there is a number L, which is to be defined by an approximation procedure using a suitable class of approximations. If a margin of error or tolerance, ϵ, is given, then the existence of a means of control, δ, must be established which is a criterion for choosing approximations to L within the given tolerance.

The epsilon-delta language is introduced even though the symbol δ does not appear. The error tolerance is brought to the fore: our goal is to stay within this tolerance. The means of controlling the error are previewed in the exercises.

Solutions Exercises 2-2

1. Find the slope of $y = f(x) = x^2 - 1$ at the point $(3,8)$ by constructing a table of values of $r(x)$ for x successively closer to 3. Verify that your answer is the limit of the slopes of the chords.

$$y = f(x) = x^2 - 1$$

$$r(x) = \frac{y - 8}{x - 3} = \frac{(x^2 - 1) - 8}{x - 3} = \left(\frac{x - 3}{x - 3}\right)(x + 3) .$$

x	2.9	3.1	2.99	3.01	3.001
$r(x)$	5.9	6.1	5.99	6.01	6.001

The values of $r(x)$ are successively closer to 6 as x is successively closer to 3. This suggests that $m = 6$. Consider any prescribed tolerance $\epsilon > 0$.

$$\begin{aligned}|r(x) - 6| &= \left|\left(\frac{x - 3}{x - 3}\right)(x + 3) - 6\right| \\ &= |(x + 3) - 6| \\ &= |x - 3| , \text{ for } x \neq 3 .\end{aligned}$$

We can guarantee that for any $\epsilon > 0$, $|r(x) - 6| = |x - 3| < \epsilon$ if x is at a distance less than ϵ from 3, or equivalently, $0 < |x - 3| < \epsilon$.

2. For the function given by $f(x) = x^2 - x + 1$, tabulate the slopes of the chords joining $\left(\frac{5}{7}, f(\frac{5}{7})\right)$ to $(x, f(x))$ for $x = \frac{5}{7} + \frac{1}{10}, \frac{5}{7} - \frac{1}{10}$, $\frac{5}{7} + \frac{1}{100}, \frac{5}{7} - \frac{1}{100}, \frac{5}{7} + \frac{1}{1000}, \frac{5}{7} - \frac{1}{1000}$, etc., as far as your time, energies, and inclinations permit. Can you predict the limit of the approximations from an inspection of the table?

$$f(x) = x^2 - x + 1$$

$$r(x) = \frac{x^2 - x + 1 - \left[\left(\frac{5}{7}\right)^2 - \left(\frac{5}{7}\right) + 1\right]}{x - \frac{5}{7}} = x - \frac{2}{7}, \; x \neq \frac{5}{7}$$

x	$\frac{5}{7} + \frac{1}{10}$	$\frac{5}{7} - \frac{1}{10}$	$\frac{5}{7} + \frac{1}{100}$	$\frac{5}{7} - \frac{1}{100}$	$\frac{5}{7} + \frac{1}{1000}$	$\frac{5}{7} - \frac{1}{1000}$
r(x)	$\frac{37}{70}$	$\frac{23}{70}$	$\frac{307}{700}$	$\frac{293}{700}$	$\frac{3007}{7000}$	$\frac{2993}{7000}$

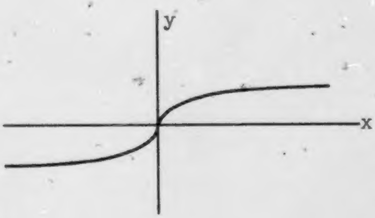
The limit of the approximations is $\frac{3}{7}$.

3. Each of the following curves passes through the origin. For each curve construct a table listing the slopes of chords with one endpoint at the origin and the other at nearby points, $(x, f(x))$. As these points are taken successively closer to the origin (e.g., $|x| < 0.1$, $|x| < 0.01$), what information do you obtain about the slope of the curve? In your opinion, is it possible to define the slope of the graph at the origin? If so, what is the slope? If not, justify your answer.

(a) $y = f(x) = x^{1/3}$

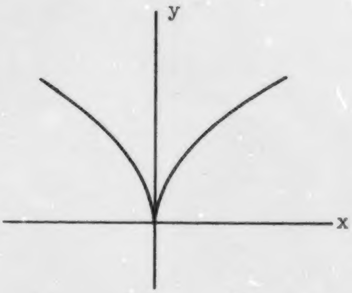
x	$\pm \frac{1}{8}$	$\pm$ 0.001	$\pm$ 0.000001
r(x)	4	100	10,000

As points are taken successively closer to the origin $r(x)$ becomes larger without bound. The slopes of the chords with one end point at the origin do not approach any number as a limit. Therefore it is not possible to define the slope of the curve at the origin.



(b) $y = f(x) = x^{2/3}$

x	$\frac{1}{8}$	$-\frac{1}{8}$	0.001	-0.001	-0.000001
$r(x)$	2	-2	10	-10	-100



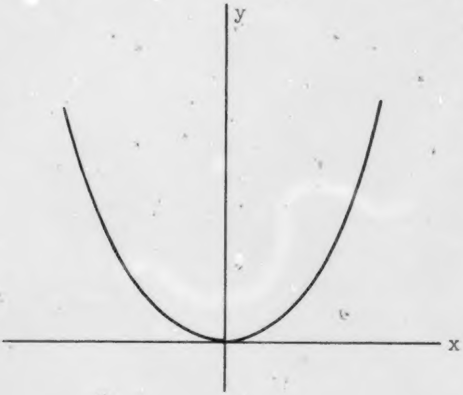
Consider points taken successively closer to the origin but remaining to the right of the origin: the slopes of chords through the origin and these points become larger without bound.

For points to the left of the origin taken successively closer to the origin, the slopes of the chords are negative and the absolute value of these slopes become larger without bound.

The slopes of these chords approach no number, for such a number would have a large absolute value and would be positive and negative at the same time.

(c) $y = f(x) = |x|^{3/2}$

x	-0.25	0.25	-0.01	0.01	0.0001
$f(x)$	0.125	0.125	0.001	0.001	0.000001
$r(x)$	-0.5	0.5	-0.1	0.1	0.01



As points are taken successively closer to the origin, the slopes of chords through the origin and these points get successively closer to zero. We are led to define the slope of the graph at the origin to be 0 .

4. (a) At each of the points (1,7) and (2,16), find the slope of $y = g(x) = 3x^2 + 4$ by constructing a table of values; then verify that your answer is the limit of the slopes of chords.

x	1.25	0.75	1.1	0.9	1.01	0.99
r(x)	6.75	5.25	6.3	5.7	6.03	5.97

The slope is 6 at the point (1,7).

Verification:

For any $\epsilon > 0$ we must have $|r(x) - 6| < \epsilon$ for x sufficiently close to 1.

$$r(x) = \frac{3x^2 + 4 - 7}{x - 1} = \frac{3x^2 - 3}{x - 1} = \frac{3(x - 1)(x + 1)}{x - 1}$$

So $r(x) = 3x + 3$, $x \neq 1$.

Then $|r(x) - 6| = |3x - 3| = 3|x - 1|$.

We see that taking the distance between x and 1 smaller than $\frac{\epsilon}{3}$, the inequality $|r(x) - 6| < \epsilon$ holds.

x	2.1	1.9	2.01	1.99	2.001	1.999
r(x)	12.3	11.7	12.03	11.97	12.003	11.997

The slope is 12 at the point (2,16).

Verification:

For any $\epsilon > 0$ we must have $|r(x) - 12| < \epsilon$ for x sufficiently close to 2.

$$r(x) = \frac{3x^2 + 4 - 16}{x - 2} = \frac{3x^2 - 12}{x - 2} = \frac{(3x + 6)(x - 2)}{x - 2}$$

So $r(x) = 3x + 6$, $x \neq 2$.

Then $|r(x) - 12| = |3x - 6| = 3|x - 2|$. Taking $|x - 2| < \frac{\epsilon}{3}$ gives $|r(x) - 12| < \epsilon$.

(b) Find the slope of $y = g(x)$ at the point (a,b) on its graph, where $b = g(a)$.

$$y = g(x) = 3x^2 + 4 \qquad b = g(a) = 3a^2 + 4$$

$$r(x) = \frac{3x^2 + 4 - b}{x - a} = \frac{3x^2 + 4 - (3a^2 + 4)}{x - a} = 3(x + a)\frac{(x - a)}{(x - a)}$$

$$= 3(x + a)\ ,\ x \neq a$$

The slope of $y = g(x)$ at the point $(a, g(a))$ is $6a$.

(c) Find the lowest point on the graph of g by methods of coordinate geometry.

The lowest point on the graph of $g(x) = 3x^2 + 4$ is at $(0,4)$, since any positive or negative values of x make $3x^2$ positive.

(d) Check your answer to (c) by using the result of (b). (If a hint is needed, one may be found in Section 1-1.)

From (c) the lowest point is $(0,4)$. From (b) the slope is zero if $a = 0$. This is at the point $(0,g(0))$ or $(0,4)$.

5. (a) Find the slope of $y = x^3$ at (a,b) where $b = a^3$. (If a hint for the simplification of $r(x)$ is needed, one may be found in Section 1-1.)

$$y = x^3$$

$$r(x) = \frac{x^3 - a^3}{x - a} = (x^2 + ax + a^2)\frac{(x - a)}{x - a}$$

$$r(x) = x^2 + ax + a^2 , \; x \neq a$$

The slope of $y = x^3$ at $x = a$ is $3a^2$.

(b) Is there any lowest point on the graph of $y = x^3$? Is there any highest point? Is there any point where the graph is horizontal?

There is no highest point or lowest point on the graph of $y = x^3$. The graph is horizontal if the slope $3a^2 = 0$, i.e., if $a = 0$.

6. What is the relationship between the slopes of the function in Number 5 corresponding to the points $x = a$ and $x = -a$? Interpret this result graphically. Give examples of other functions having this property.

They are equal. The tangents have the same slope and are parallel. Other functions having this property are polynomial functions of the form

$$P(x) = a\,x^{2n+1} + b\,x^{2n-1} + \dots + h\,x^3 + k\,x\,; \quad n\text{, an integer.}$$

[Accept from students any specific examples, then ask them to check the sum of any two of these to generalize to obtain above polynomial. Graphically, this curve exhibits symmetry with respect to the origin. The function $y = f(x) = x^3 + 3$ would have the same properties except that it is not symmetric with respect to the origin, but it is symmetric with respect to the point $(0,3)$. We say it is centrosymmetric with respect to this point. In the next section, there will be a problem about the centrosymmetric properties of a general cubic and a word now will make that easier.

The result also holds for curves whose equation is $f(x,y) = 0$ where the equation remains the same if x is changed to $-x$ and y to $-y$, i.e., $f(x,y) = f(-x,-y)$. Such a curve is said to be centrosymmetric about the origin $(0,0)$. For example: $x^2 + y^2 = a^2$; $xy = a$.]

7. What is the relationship between the slopes of the function in Number 4 corresponding to the points, $x = a$ and $x = -a$? Interpret this result graphically. Give examples of other functions having this property.

Here $r(a) = -r(-a)$. This means the angles of inclination of the graph at these points are supplementary. This also produces a form of symmetry (with respect to the y-axis). As in Number 6, functions would be generally described as polynomials, but here only even exponents for x would be found.

The result also holds for curves whose equation is $f(x,y) = 0$ where the equation remains the same if x is changed to $-x$, i.e., $f(x,y) = f(-x,y)$ [symmetry with respect to the y-axis]. For example,

$$x^2 + y^2 = 2\ ,\ \sin^2 x + \sin y = 1\ .$$

8. (a) Find the slope of the graph of $h : x \rightarrow 4x^3 - 3x^2$ at (a,b), where $b = h(a)$.

$$r(x) = \frac{4x^3 - 3x^2 - (4a^3 - 3a^2)}{x - a}$$

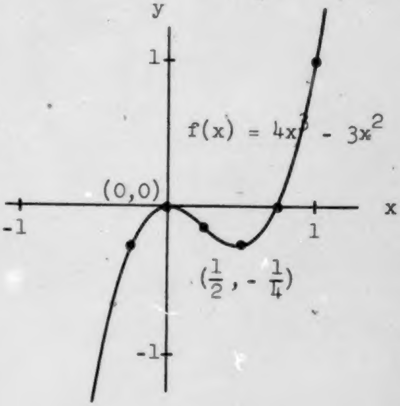
$$= 4(x^2 + ax + a^2) - 3(x + a)\ ,\ x \neq a)$$

The slope of the graph at the point $\big(a,h(a)\big)$ is

$$12a^2 - 6a\ .$$

(b) Find all points where the graph of h is horizontal. Can you characterize these points as "highest" or "lowest," perhaps in a restricted sense?

The graph is horizontal if $12a^2 - 6a = 0$, i.e., if $a = 0$ or $a = \frac{1}{2}$. At the point $(0,0)$ the graph has a "local highest" point: at the point $\left(\frac{1}{2}, -\frac{1}{4}\right)$ the graph has a "local lowest" point.



Solutions Exercises 2-3

1. Find the slope for $x = a$ of the general linear function $f : x \longrightarrow Ax + B$ (where A and B are any constants except that $A \neq 0$) and compare your result to that obtained from the standard slope-intercept form of the equation of a straight line in coordinate geometry.

$$f(x) = Ax + B , \qquad A \neq 0$$

$$r(x) = \frac{Ax + B - (Aa + B)}{x - a} = A , \; x \neq a$$

The standard slope-intercept form of the equation of a straight line in coordinate geometry is $y = mx + b$, where m is the slope. Then for $y = Ax + B$, the slope-intercept form also gives $m = A$.

2. For what values of k does the line $y = k$ intersect the parabola

$$y = Ax^2 + Bx + C \quad (A \neq 0)$$

in

(a) no points?
(b) 1 point?
(c) 2 points?
(d) What is the lowest or highest point of the given parabola?

The number of points of intersection of $y = k$ with $y = Ax^2 + Bx + C$ will correspond to the number of real solutions of

$$Ax^2 + Bx + C - k = 0 .$$

The discriminant of $Ax^2 + Bx + C - k = 0$ is $B^2 - 4A(C - k)$ and this equals zero when

(b) $k = C - \frac{B^2}{4A}$ and the line intersects the parabola at one point.

For (a) and (c), the values for k depend upon the sign of A. If $A > 0$, then

(a) if $k < C - \frac{B^2}{4A}$ the line does not intersect the parabola.

(c) if $k > C - \frac{B^2}{4A}$ the line intersects the parabola at two distinct points.

Reverse the above inequalities if $A < 0$.

(d) The lowest $(A > 0)$ or highest $(A < 0)$ point of the parabola is $\left(-\frac{B}{2A}, C - \frac{B^2}{4A}\right)$.

3. (a) Find the highest point on the graph of

$$g(x) = 5 - 6x - x^2$$

using Number 2.

The highest point is $(-3,14)$.

(b) Explain geometrically why the point in (a) can also be obtained by finding where the slope of $g(x)$ is zero.

If the slope of $g(x)$ is 0 at $x = a$, the equation of the horizontal line described in Number 2 is $y = g(a) = k$. This line intersects the curve at only one point, the highest point.

4. (a) What is the greatest possible number of points where the graph of a quadratic function $Ax^2 + Bx + C$ may be horizontal?

The graph of a quadratic function $x \longrightarrow Ax^2 + Bx + C$, $A \neq 0$, is horizontal at points where the slope m is zero.

$$m = 2Aa + B = 0$$

So $a = \frac{-B}{2A}$ is the only point where the graph of the quadratic is horizontal.

(b) Is it possible for the graph to be horizontal at less than the maximum number of points, or nowhere horizontal? If the answer to either question is affirmative, given an example (in the form of a specific function).

The graph of the quadratic function always has one horizontal point.

5. (a) Given $y = f(x) = 20x - 3x^2$. Find the slope of the curve at the point (a,b) , where $b = f(a)$.

$$r(x) = \frac{20x - 3x^2 - (20a - 3a^2)}{x - a} = 20 - 3(x + a) \quad , \quad x \neq a$$

At the point $(a,f(a))$, the slope is $20 - 6a$.

(b) Where is the slope zero? How can you use this information in plotting the graph of f ?

The slope is zero where $20 - 6a = 0$, i.e., at $a = \frac{10}{3}$. The highest point on the graph is at $\left(\frac{10}{3}, f\left(\frac{10}{3}\right)\right)$.

6. (a) Find the slope of the curve with equation $y = h(x) = Ax^3 + Bx^2 + Cx + D$ (the graph of the general cubic function) at (a,b) , where $b = h(a)$; here A , B , C , D are any constants, except that $A \neq 0$. (If you need a hint for the simplification of $r(x)$, it may be found in Section 1-1.)

$$r(x) = \frac{A(x^3 - a^3) + B(x^2 - a^2) + C(x - a)}{x - a}$$

$$= A(x^2 + ax + a^2) + B(x + a) + C \quad , \quad x \neq a$$

The slope of y at $(a,h(a))$ is $3Aa^2 + 2Ba + C$.

(b) What is the greatest possible number of points where the graph of cubic function h may be horizontal?

The greatest number of points where the graph of a cubic function may be horizontal is 2. The quadratic equation

$$3Aa^2 + 2Ba + C = 0$$

has at most two real roots.

(c) Is it possible for a cubic function to have its graph horizontal at less than the maximum number of points? If the answer is "Yes," give an example of such a function.

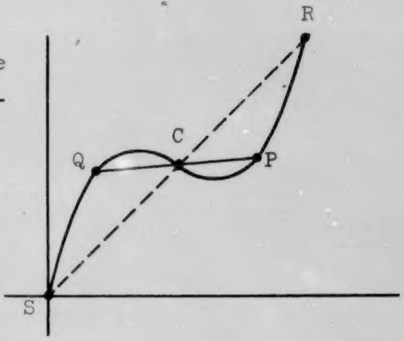
A cubic may fail to have its graph horizontal at two points if the discriminant of the above quadratic in a is not positive. If $4B^2 - 12AC < 0$, then there are no points where the graph of the cubic has a horizontal slope. If $4B^2 - 12AC = 0$, or $B^2 = 3AC$, then there is one point of the cubic having a horizontal slope.

(d) Is it possible for a cubic function to have its graph nowhere horizontal? If the answer is "Yes," give an example of such a function.

Yes, see (c) above. If $B = 0$ and $A > 0$ and $C > 0$, we can make many examples such as

$$x^3 + 8x = 0 .$$

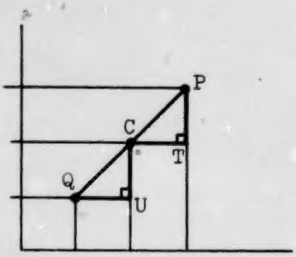
Λ 7. Show that the curve of Exercise 6 is centrosymmetric about the point $\left(-\frac{B}{3A}, h\left(-\frac{B}{3A}\right)\right)$.



Centrosymmetric means that given the above point as a center, a line segment starting at any other point on the curve going through the center and extended will intersect the curve again so that the center is the midpoint of this segment.

$PC = CQ$ and

$RC = CS$



It is easily shown that if PC = CQ and if PQ is a straight line, then QU = CT and PT = CU . So if we take values of x the same distance in each direction from $-\frac{B}{3A}$ we should get y values which differ from $h(-\frac{B}{3A})$ by the same amount.

Let us take $x_1 = -\frac{B}{3A} - E$, $x_2 = -\frac{B}{3A}$, $x_3 = -\frac{B}{3A} + E$ and compute $h(x)$ in each case.

$$h(x_3) = A(-\frac{B}{3A} + E)^3 + B(-\frac{B}{3A} + E)^2 + C(-\frac{B}{3A} + E) + D$$

$$h(x_2) = A(-\frac{B}{3A})^3 + B(-\frac{B}{3A})^2 + C(-\frac{B}{3A}) + D$$

$$h(x_1) = A(-\frac{B}{3A} - E)^3 + B(-\frac{B}{3A} - E)^2 + C(-\frac{B}{3A} - E) + D$$

and then:

$$h(x_3) - h(x_2) = A(\frac{B^2E}{3A^2} - \frac{BE^2}{A} + E^3) + B(-\frac{2BE}{3A} + E^2) + CE = -\frac{B^2E}{3A} + AE^3 + CE$$

$$h(x_2) - h(x_1) = A(\frac{B^2E}{3A^2} + \frac{BE^2}{A} + E^3) + B(-\frac{2BE}{3A} - E^2) + CE = -\frac{B^2E}{3A} + AE^3 + CE .$$

So

$$h(x_3) - h(x_2) = h(x_2) - h(x_1) .$$

Of course, if students have studied Analytic Geometry and have had translations (SMSG--Analytic Geometry, Chapter 10), translating the equation so that the center of symmetry becomes the origin gives:

$$Y = AX^3 + EX$$

which is referred to in No. 6 of Exercises 2-2.

8. A function h is defined by the formulas

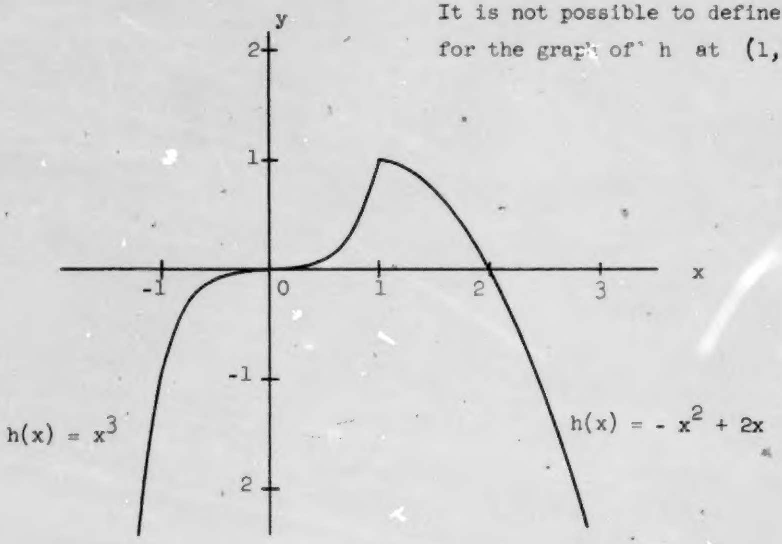
$$h(x) = \begin{cases} x^3 , & x < 1 , \\ -x^2 + 2x , & x \geq 1 . \end{cases}$$

Find the slope of the graph of h for $x < 1$ and for $x > 1$. Is it possible in your opinion to define a slope for the graph at $(1,1)$? Give an argument to support your answer. (A sketch may be helpful in answering the question.)

For $x < 1$, $h(x) = x^3$ and $m = 3a^2$;

for $x \geq 1$, $h(x) = -x^2 + 2x$ and $m = -2a + 2$.

It is not possible to define a slope for the graph of h at $(1,1)$.



9. Until now our discussion of the idea of direction and slope for a curve has been generally at a theoretical level. Although we know from Section 1-1 that the concept of slope will ultimately be useful in "best value" problems it is satisfying to have another more immediate application. You are probably familiar with the fact that large telescopes and automobile headlights use parabolic mirrors. A parabolic reflector can bring a bundle of parallel rays like those from a star* to a sharp focus. You are now able to demonstrate the sharp focusing property of the parabola.

According to Heron's Law of reflection for a ray of light incident upon a smooth mirror, the incident ray and the reflected ray make equal angles with the mirror. Suppose the shape of the cross section through the axis of the mirror is given by the graph of $y = x^2$. Prove that all incident rays parallel to the y-axis that the reflected rays have a common point of intersection as in the figure. This common point is called the focus of the parabola. It can also be shown that this property characterizes the parabola; i.e., if the curve is such that all parallel rays pass through a common point after reflection, then the curve must be a parabola.

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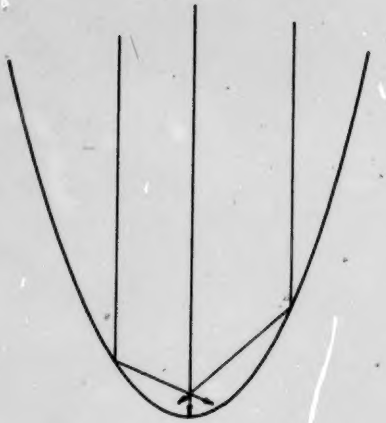
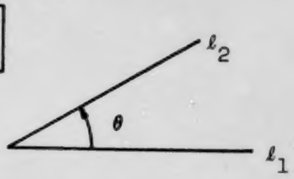
*At stellar distances the deviation from parallelism of all rays reaching the earth from a given star is utterly negligible.



This old chestnut appears here to enable the teacher to discuss equations of lines (focus and point of reflection determine line) and angle between two lines. Discuss reflection on a plane mirror and if physics class has not anticipated this, indicate that angle of incidence equals angle of reflection. Angles between lines require the theorem:

$$\theta = \text{arc tan}\left[\frac{m_2 - m_1}{1 + m_1 m_2}\right]$$



where m_1 is the slope of ℓ_1 and
m_2 is the slope of ℓ_2.

Here, the plane is replaced by the tangent line, which has the slope of the curve at that point.

Do not lose time trying to have students complete the solution. It is here for motivation not exhaustion. The converse involves differential equations, and this idea may be reached this year by the class.

Solutions Exercises 2-4

1. [NOTE: Even though this problem consists of many parts, it doesn't take very long to do. Also, one does not have to assign all the parts. It is included because there are very few problems in which physical considerations are stressed.]

Let us assume that a pellet is projected straight up and after awhile comes straight down via the same vertical path to the place on the ground from which it was launched. After t seconds the pellet is s feet above the ground. Some of the ordered pairs (t,s) are given in the following table.

t	0	1	2	3	4	5	6	7	8	9	10
s	0	144	256	336	384	400	384	336	256	144	0

We shall intentionally avoid certain physical considerations such as air resistance. Moreover, we shall deal with simple numbers rather than quantities measured to some prescribed degree of accuracy which might arise from the data of an actual projectile problem in engineering.

(a) Interpolate from the data given to determine the height of the projectile after eight and nine seconds respectively. (Guess, using symmetry as your guide.) Does extrapolation to find values of s for $t = -1$ or $t = 11$ make sense on physical grounds? After how many seconds does the projectile appear to have reached its maximum height? What seems to be the maximum height?

For 8 seconds, $s = 256$ feet. For 9 seconds, $s = 144$ feet. At $t = -1$, or $t = 11$, s is negative and may be interpreted as below the surface of the ground. The maximum height seems to be 400 ft. reached after 5 seconds.

(b) Does s appear to be a function of t ? If so, discuss the domain and range, taking physical considerations into account.

A table as shown where there is a unique number for s under each t represents a function. The domain is $0 \leq t \leq 10$ or the interval $[0,10]$. The range is $0 \leq s \leq 400$ or the interval $[0 , 400]$.

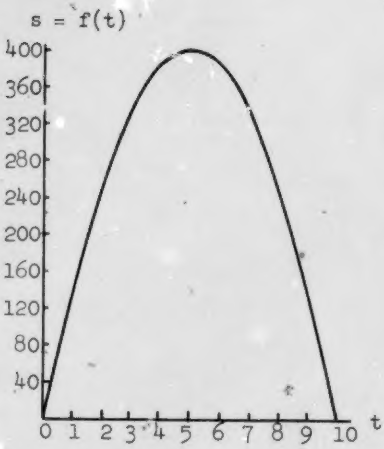
(c) If we were to plot a graph of $s = f(t)$,

(1) is it plausible on physical grounds to restrict our graph to the first quadrant? YES.

(2) Does the data suggest that the scale on the s-axis (vertical) should be the same as the scale on the t-axis (horizontal?) NO.

(d) Keeping in mind your responses to part (c), plot the ordered pairs (t,s) from the table. Connect the points with a smooth curve. What is the name of the function suggested by the graph? (Parabola) On physical grounds is it feasible that there would be a real value of s for every real number assigned to t over the interval $0 \leq t \leq 10$? Were we probably justified in connecting the points?

Every real number for t over the interval $0 \leq t \leq 10$, would produce a value for s, so we were justified in connecting the points.



(e) Assuming that the equation $s = f(t) = At^2 + Bt + C$ was used to develop the entries in our table, find values for constants A, B, and C.

$$s = f(t) = At^2 + Bt + C$$

For $t = 0$ $\quad f(0) = C = 0$

$t = 1$ $\quad f(1) = A + B = 144$

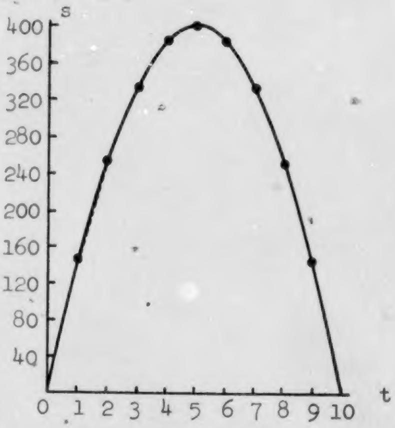
$t = 2$ $\quad f(2) = 4A + 2B = 256$

$$\left.\begin{matrix} A + B = 144 \\ 4A + 2B = 256 \end{matrix}\right\} \quad \begin{matrix} A = -16 \\ B = 160 \end{matrix}$$

$A = -16$, $B = 160$, and $C = 0$

(f) Sketch the graph given by the equation $s = 160t - 16t^2$ over the interval $0 \leq t \leq 10$. Using a more carefully plotted graph of the above set, connect the point where $t = 1$ with the point where $t = 2$ with a chord. What is the slope of this chord? Estimate the slope of the curve at $t = 1$ and $t = 2$.

Slope of chord through points $(1, 144)$ and $(2, 256) = 112$. At $t = 1$, it would be a little larger (steeper) than 112; at $t = 2$, slightly smaller than 112.

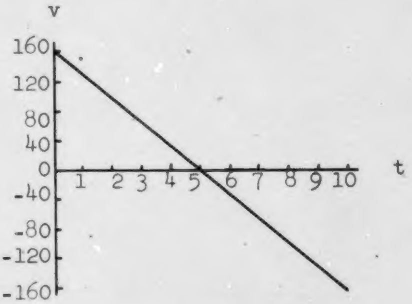


(g) If the units of s are feet and the units of t are seconds, what are the units of slope? What word is commonly associated with this ratio of units? What would you guess are the physical interpretations of positive, zero, and negative values of this ratio?

Slope is measured in units of feet per second. Speed is the usual name. Positive values would indicate upward movement; zero, a resting position; while negative values would indicate a falling or downward movement.

(h) Draw the graph of $v = 160 - 32t$ over the interval $0 \leq t \leq 10$. Compare the values of v for $t = 1$ and $t = 2$ respectively with your estimates for the slopes of the graph of $s = 160t - 16t^2$ in part (f).

At $t = 1$, $v = 128$ and at $t = 2$, $v = 96$.



(i) Average the values of v for $t = 1$ and $t = 2$ and compare this average with the slope of the chord connecting the points where $t = 1$ and $t = 2$ in part (f). 112 ft/sec.

(j) If the units of v are ft/sec and the units of t are seconds, what are the units of the slope of the line $v = 160 - 32t$? What word from physics is commonly associated with this ratio of units? Does the minus sign along with the particular numerical value of this slope have any special connotation from your experience?

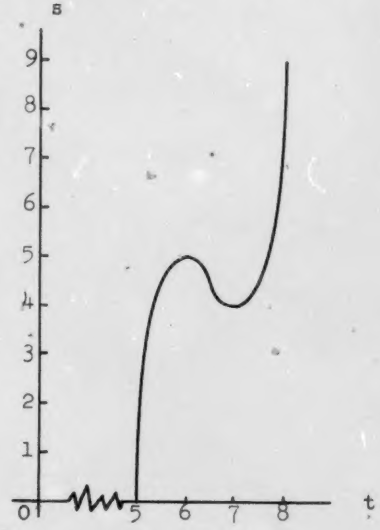
The slope of the line $v = 160 - 32t$ is in units of feet per second per second. The word in physics is acceleration. -32 refers (in feet per second per second (ft./sec.^{2})) to the acceleration due to the force of gravity at sea level.

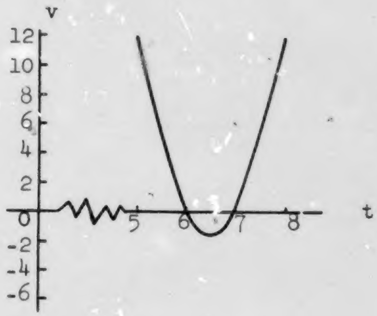
2. (a) Derive the velocity function for the motion as given in Example 2-4a.

$$s = \phi(t) = 2t^3 - 39t^2 + 252t - 535$$

$$v = 6t^2 - 78t + 252$$

(b) Sketch the graph of $s = \phi(t)$, (called the world line) and the v vs. t curve (i.e., the graph of the velocity as a function of time).





(c) Compare the time when s equals a maximum or a minimum and when velocity $v = 0$. Explain this physically.

The times are the same. Non-zero velocities indicate a change taking place in a definite direction. To shift directions "smoothly" it must come to rest, or go through a velocity of zero.

(d) Given only that $\phi(6) = 5$ for the function ϕ that describes the motion, show that there is a second time t when $\phi(t) = 5$, and find that value of t. (This is not done by calculus.)

$$5 = 2t^3 - 39t^2 + 252t - 535$$

Solving a cubic usually is difficult, but we know that 6 is a root which enables us to reduce the cubic to a quadratic.

$$2t^3 - 39t^2 + 252t - 540 = 0$$

$(t - 6)(2t^2 - 27t + 90) = 0$, which may be factored further to

$(t - 6)^2(2t - 15) = 0$, therefore $t = 6$ or $t = 7\frac{1}{2}$.

(e) Find the time of greatest speed between $t = 6$ and $t = 7$.

From the v vs. t graph, it appears that the absolute value of the velocity is greatest at $t = 6\frac{1}{2}$ which is the time of greatest speed.

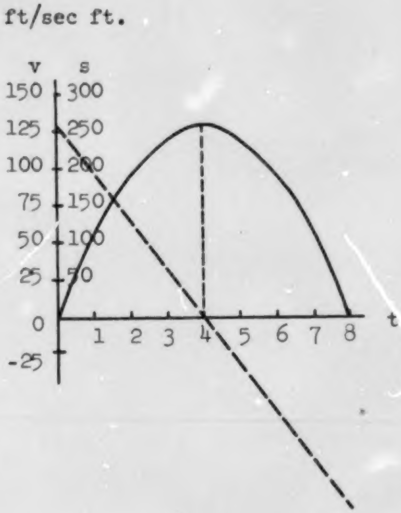
3. Find the velocity of an object whose location along a straight line is described by the equation $s = 128t - 16t^2$. Sketch the curves of s vs. t and v vs. t on the same set of axes.

(a) During what time interval or intervals is the object moving toward the location $s = 0$?

$4 < t < 8$

(b) What are the values of v and t when s is a maximum?

$v = 0$ and $t = 4$



4. A ball is thrown upward with a velocity of 32 feet per second. Its height h in feet after t seconds is described by the equation $h = 32t - 16t^2$.

(a) What is the velocity of the ball when its height first reaches 12 feet? When it again reaches 12 feet?

$$h = 32t - 16t^2 \qquad\qquad v = 32 - 32t$$

$$h = 12 = 32t - 16t^2 \qquad \text{or}$$

$$0 = 16t^2 - 32t + 12$$

$$= 4t^2 - 8t + 3$$

$$= (2t - 3)(2t - 1)$$

which is true when $t = \frac{1}{2}$ or $\frac{3}{2}$. The first time it reaches 12 feet is $\frac{1}{2}$ second after it is thrown; its velocity at that time is $v = 32 - 32(\frac{1}{2}) = 16$. The second time, $t = \frac{3}{2}$, at which time $v = 32 - 32(\frac{3}{2}) = -16$.

(b) How high does it go, and at what time does the ball reach its highest position?

$$v = 0 = 32 - 32t \text{ , at } t = 1 \text{ .}$$

One second after being thrown it reaches its highest point, 16 feet.

5. An object is projected up a smooth inclined plane in a straight line. Its distance s in ft. from the starting point after t seconds is described by the equation $s = 64t - 8t^2$. After the object reaches its highest point it slides back along its original path to the starting point according to the equation $\bar{s} = 8(t - t_n)^2$. Here $\bar{s}$ is the distance of the object from the highest point and t_n is the time it took the object to reach the highest point.

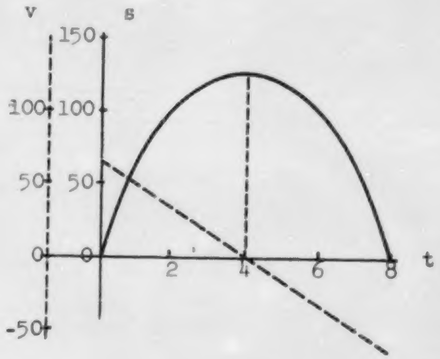
(a) Determine how long it took for the object to make the up and down trip.

$s = 64t - 8t^2$ and maximum $s = 128$ since $m = 64 - 16t$ and $t = 4$ for maximum s . Since $t_n = 4$, $\bar{s} = 8(t - 4)^2$ $= 8t^2 - 64t + 128$. But $\bar{s}$ is the distance from 128 back toward zero and so we may write:

$$s = 128 - \bar{s} = 64t - 8t^2 \quad \text{for} \quad t \geq 4 \text{ .}$$

Since the upward travel is described by $s = 64t - 8t^2$ for $0 \leq t \leq 4$, we have for the entire trip the function $s = 64t - 8t^2$, $0 \leq t \leq 8$ since s again equals zero when $t = 8$.

(b) Sketch the s vs. t curve for the up and down motion using one set of coordinates. Do likewise for the v vs. t curve.



6. **The location of an object on a straight line is given by the formula $s = pt^2 + qt + r$, where p, q, and r are real constants. Find all instants of time when the object is at rest, and show how the number of such instants depends on the constants p, q, and r.**

From $\qquad s = \phi(t) = pt^2 + qt + r$

we get $\qquad v_0 = 2pt_0 + q$.

The object is at rest where $2pt_0 + q = 0$ or $t_0 = -\frac{q}{2p}$. The time $t_0 = -\frac{q}{2p}$ is dependent on q and p and must be unique since q and p are real numbers.

7. **For any but the very simplest motions, the function ϕ describing the location at time t is unlikely to be expressible in terms of a single formula for the entire duration of the motion. Here is a more plausible description of a motion:**

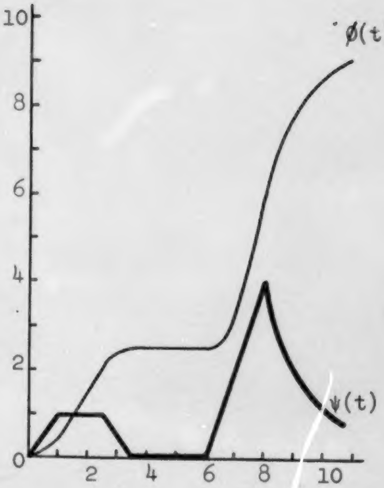
$$\phi(t) = \begin{cases} \frac{t^2}{2}\,, & 0 \leq t \leq 1 \\ t - \frac{1}{2}\,, & 1 \leq t \leq \frac{5}{2} \\ \frac{-t^2}{2} + \frac{7t}{2} - \frac{29}{8}\,, & \frac{5}{2} \leq t \leq \frac{7}{2} \\ \frac{5}{2}\,, & \frac{7}{2} \leq t \leq 6 \\ t^2 - 12t + \frac{77}{2}\,, & 6 \leq t \leq 8 \\ \frac{21}{2} - 4(t-7)^{-1}\,. & 8 \leq t \leq 10 \end{cases}$$

(a) **Compute the function $\psi(t)$ that describes the velocity of the motion with location $s = \phi(t)$.**

$$\psi(t) = \begin{cases} t\,, & 0 \leq t \leq 1 \\ 1\,, & 1 \leq t \leq \frac{5}{2} \\ -t + \frac{7}{2}\,, & \frac{5}{2} \leq t \leq \frac{7}{2} \\ 0\,, & \frac{7}{2} \leq t \leq 6 \\ 2t - 12\,, & 6 \leq t \leq 8 \\ \frac{4}{(t-7)^2}\,. & 8 \leq t \leq 10 \end{cases}$$

(b) It is claimed that $s = \phi(t)$ and $v = \psi(t)$ are functions. What has to be checked to verify this? Does it check? Show the graph of each of these functions on the same axes.

A function must have one value in its range for each number in its domain. The statement of the problem included the endpoint of each interval in both intervals. For each of these endpoints, we do get the same image in both intervals (for both $\phi(t)$ and $\psi(t)$.) We can see this on the graph.



(c) During what time intervals do you think the speed of the motion is increasing? decreasing?

The velocity is increasing during the intervals

$$0 < t < 1 \quad \text{and} \quad 6 < t < 8 ,$$

and decreasing during the intervals

$$\frac{5}{2} < t < \frac{7}{2} \quad \text{and} \quad 8 < t < 10 .$$

(d) Does the object spend any time between $t = 0$ and $t = 10$ standing still? Does it have any other instants of zero velocity during the motion?

The object stands still during the interval

$$\frac{7}{2} \leq t \leq 6 .$$

The velocity is zero at the start, i.e., at $t = 0$.

Solutions Exercises 2-5

1. Find the derivative of f at a for $f : x \longrightarrow f(x)$, where $f(x)$ equals each of the following.

(a) $f(x) = \frac{1}{x + 1}$

$$r(x) = \frac{\frac{1}{x+1} - \frac{1}{a+1}}{x - a} = \frac{-1}{(x+1)(a+1)}, \quad x \neq a$$

$$m = \frac{-1}{(a+1)^2}$$

(b) $f(x) = \frac{1}{Ax + B}$, where A and B are constants, $A \neq 0$

$$r(x) = \frac{\frac{1}{Ax+B} - \frac{1}{Aa+B}}{x - a} = \frac{-A}{(Ax+B)(Aa+B)}, \quad x \neq a$$

$$m = \frac{-A}{(Aa+B)^2}$$

(c) $f(x) = \frac{1}{x^2}$

$$r(x) = \frac{\frac{1}{x^2} - \frac{1}{a^2}}{x - a} = \frac{-(x+a)}{x^2 \cdot a^2}, \quad x \neq a$$

$$m = \frac{-2a}{a^4} = \frac{-2}{a^3}$$

(d) $f(x) = \frac{1}{x^3}$

$$r(x) = \frac{\frac{1}{x^3} - \frac{1}{a^3}}{x - a} = \frac{-(x^2 + ax + a^2)}{x^3 a^3}, \quad x \neq a$$

$$m = \frac{-3a^2}{a^6} = \frac{-3}{a^4}$$

(e) $f(x) = \dfrac{1}{\sqrt{x}}$

$$r(x) = \frac{\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{a}}}{x - a} = \frac{-1}{\sqrt{x}\,\sqrt{a}\,(\sqrt{x} + \sqrt{a})}\,,\ x \neq a$$

$$m = \frac{-1}{2a\,\sqrt{a}}$$

(f) $f(x) = x^{3/2}$

$$r(x) = \frac{x^{3/2} - a^{3/2}}{x - a} = \frac{(x^{1/2} - a^{1/2})(x + x^{1/2}a^{1/2} + a)}{x - a} = \frac{x + x^{1/2}a^{1/2} + a}{x^{1/2} + a^{1/2}}\,,$$

$$x \neq a$$

$$m = \frac{3}{2}\,a^{1/2}$$

2. For each of the functions of Exercise 1, and also for the functions $g(x) = \dfrac{1}{x}$ and $h(x) = \sqrt{x}$ of the text, list the following: the domain of the function; all points of horizontal slope (if any); the highest point on the graph of the function (if any); the lowest point on the graph of the function (if any).

		domain	horizontal points	highest point	lowest point
	$g(x) = \frac{1}{x}$	$x \neq 0$	none	——	——
	$h(x) = \sqrt{x}$	$x \geq 0$	none	——	$x = 0$
(1-a)	$f(x) = \frac{1}{x + 1}$	$x \neq -1$	none	——	——
(1-b)	$f(x) = \frac{1}{Ax + B}$	$x \neq \frac{-B}{A}$	none	——	——
(1-c)	$f(x) = \frac{1}{x^2}$	$x \neq 0$	none	——	——
(1-d)	$f(x) = \frac{1}{x^3}$	$x \neq 0$	none	——	——
(1-e)	$f(x) = \frac{1}{\sqrt{x}}$	$x > 0$	none	——	——
(1-f)	$f(x) = x^{3/2}$	$x \geq 0$	$x = 0$	——	$x = 0$

3. Use the definition of derivative to differentiate f at both $a = -2$ and $a = 2$ if $f(x)$ equals

(a) $f(x) = (x - 2)^2$

$$m = \lim_{x \to a} \frac{(x - 2)^2 - (a - 2)^2}{x - a} = \lim_{x \to a} \left[(x - 2) + (a - 2)\right]\left[\frac{(x - 2) - (a - 2)}{x - a}\right]$$

$$= 2a - 4$$

At $a = -2$, $m = -8$

At $a = 2$, $m = 0$

(b) $f(x) = \frac{1 + x}{1 - x}$

$$m = \lim_{x \to a} \frac{\frac{1 + x}{1 - x} - \frac{1 + a}{1 - a}}{x - a} = \lim_{x \to a} \frac{2(x - a)}{(1 - x)(1 - a)(x - a)} = \frac{2}{(1 - a)^2}$$

At $a = -2$, $m = \frac{2}{9}$

At $a = 2$, $m = 2$

(c) $f(x) = 3x - x^3$

$$m = \lim_{x \to a} \frac{(3x - x^3) - (3a - a^3)}{x - a} = \lim_{x \to a} \frac{3(x - a) - (x^3 - a^3)}{x - a} = 3 - 3a^2$$

At $a = -2$, $m = -9$

At $a = 2$, $m = -9$

(d) $f(x) = x^2 - x - x^{-1}$

$$m = \lim_{x \to a} \frac{x^2 - x - x^{-1} - (a^2 - a - a^{-1})}{x - a} = \lim_{x \to a} \frac{(x^2 - a^2) - (x - a) - \left(\frac{1}{x} - \frac{1}{a}\right)}{x - a}$$

$$= 2a - 1 + \frac{1}{a^2}$$

At $a = -2$, $m = -\frac{19}{4}$

At $a = 2$, $m = \frac{13}{4}$

4. For each of the functions f whose values $f(x)$ are described below, find the derivative at a, where a is in the domain of the function.

(a) $5x - x^2$ $\qquad m = 5 - 2a$

(b) $3 - x - x^2$ $\qquad m = -1 - 2a$

(c) $x^3 - 2x$ $\qquad m = 3a^2 - 2$

(d) $\dfrac{2x}{x+1}$

$$m = \lim_{x \to a} \frac{\frac{2x}{x+1} - \frac{2a}{a+1}}{x - a} = \lim_{x \to a} \frac{2(x-a)}{(x+1)(a+1)(x-a)}$$

$$= \frac{2}{(a+1)^2}$$

(e) $\dfrac{2}{x-1}$ $\qquad m = \dfrac{-2}{(a-1)^2}$

(f) $\dfrac{x^2+x}{x^2-1}$

$$m = \lim_{x \to a} \frac{\frac{x^2+x}{x^2-1} - \frac{a^2+a}{a^2-1}}{(x-a)} = \lim_{x \to a} \frac{-(x^2-a^2) - ax(x-a) - (x-a)}{(x-a)(x^2-1)(a^2-1)}$$

$$= \frac{-2a - a^2 - 1}{(a^2-1)^2} = \frac{-1}{(a-1)^2}$$

5. By the methods of this section find the slope of the graph of each of the following functions at the point $(1,1)$.

(a) $f : x \longrightarrow x^3$

$m_1 = 3a^2$, $\therefore$ $m_1 = 3$ at the point $(1,1)$

(b) $g : x \longrightarrow x^{1/3}$

$$m_2 = \lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{x - a} = \lim_{x \to a} \frac{(x^{1/3} - a^{1/3})(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})}{(x-a)(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})}$$

$$= \frac{1}{3a^{2/3}}$$

$\therefore$ $m_2 = \frac{1}{3}$ at the point $(1,1)$

What is the relationship between the two answers? Explain this relationship.

$m_1 = \frac{1}{m_2}$. The graph of $y = f(x) = x^3$ is the reflection of the graph of $y = f(x) = x^{1/3}$ in the line $y = x$.

$f : x \longrightarrow x^3$ and $f : x \longrightarrow x^{1/3}$ are inverse functions.

6. Use a table of sines to obtain approximations to the limit of $\frac{\sin x}{x}$ as x approaches $a = 0$, and make a conjecture as to the value of this limit. What can you conclude about the slope of the graph of $y = \frac{\sin x}{x}$ at the origin?

As x approaches $a = 0$, $\frac{\sin x}{x}$ approaches 1 ; we guess that

$$\lim_{x \sim a} \frac{\sin x}{x} = 1 .$$

Note that $y = \frac{\sin x}{x}$ is not defined at $x = 0$, hence the slope of the graph is not defined at $x = 0$.

7. Write the derivative of the cosine function at $x = a$ as a limit, for $a = 0$ and $a = \frac{\pi}{2}$, and show what limits must be evaluated before the derivative can be obtained.

$$m = \lim_{x \sim 0} \frac{\cos x - \cos 0}{x - 0} = \lim_{x \sim 0} \frac{\cos x - 1}{x}$$

$$m = \lim_{x \sim \frac{\pi}{2}} \frac{\cos x - \cos \frac{\pi}{2}}{x - \frac{\pi}{2}} = \lim_{x \sim \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}}$$

8. In Chapter 8 we shall define a function $f : x \longrightarrow 2^x$ in such a way as to have all real numbers x in its domain, and also to satisfy all the familiar laws of exponents valid for 2^n when n is a rational number. Compute the ratios $r(x)$ that serve to approximate the derivative of f at $x = a = 0$, and show what limit problem must be solved before we can differentiate the function.

$f(x) = 2^x$, $r(x) = \frac{2^x - 2^a}{x - a}$, $m = \lim_{x \sim 0} \frac{2^x - 1}{x}$

9. Prove that the derivative at $x = a$ of:

(a) $g : x \longrightarrow \frac{1}{x}$ is $-\frac{1}{a^2}$. (b) $h : x \longrightarrow \sqrt{x}$ is $\frac{1}{2\sqrt{a}}$.

See Example 2-5a; for proof, see Example 3-3e.

See Example 2-5b; for proof, see Example 3-3c.

10. Find the slopes (if any exist) of the following curves at points for which $x = y$:

(a) $x + y = A$

$y = -x + A$

$m = -1$ for all x

(b) $x + xy + y = 3A$

$$y = \frac{3A - x}{x + 1}$$

$$r(x) = \frac{\frac{3A - x}{x + 1} - \frac{3A - a}{a + 1}}{x - a} = \frac{(-3A - 1)}{(x + 1)(a + 1)} \quad \text{for } x \neq a$$

$$m = \frac{-3A - 1}{(a + 1)^2}$$

At a point where $x = y$ we have $a = x = y = -1 \pm \sqrt{1 + 3A}$ and

$$m = \frac{-3A - 1}{(-1 \pm \sqrt{1 + 3A} + 1)^2} = \frac{-(3A + 1)}{3A + 1} = -1$$

(c) $xy = A^2$

$$y = \frac{A^2}{x}$$

$$r(x) = \frac{\frac{A^2}{x} - \frac{A^2}{a}}{x - a} = -\frac{A^2}{xa} \quad \text{for } x \neq a$$

$$m = -\frac{A^2}{a^2}$$

At $y = x$, $a = x = \pm A$ and $m = -1$

(d) $|x + y| + |x - y| = 2A$

The following four cases must be considered.

① $\begin{cases} x + y > 0 \\ x - y > 0 \end{cases}$

$x + y + x - y = 2A$,

$x = A$ and $|y| \leq A$

② $\begin{cases} x + y > 0 \\ x - y < 0 \end{cases}$

$x + y - x + y = 2A$,

$y = A$ and $|x| \leq A$

③ $\begin{cases} x + y < 0 \\ x - y > 0 \end{cases}$

$-x - y + x - y = 2A$,

$y = -A$ and $|x| \leq A$

④ $\begin{cases} x + y < 0 \\ x - y < 0 \end{cases}$

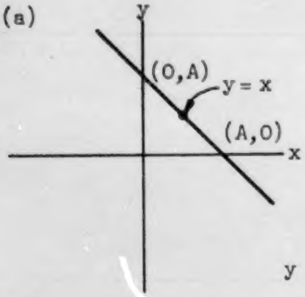
$-x - y - x + y = 2A$,

$x = -A$ and $|y| \leq A$

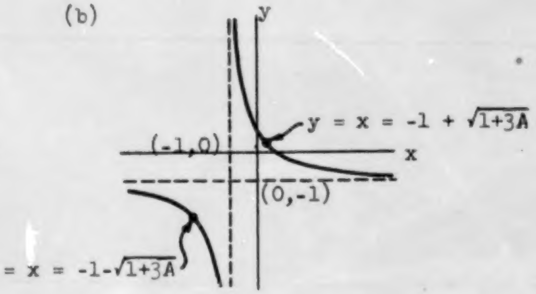
Of these 4 cases only cases 2 and 3 are functions and the slopes then are zero at every point but where $y = x$ which occurs at point (A,A) in case 2 and $(-A,-A)$ in case 3 . At these points the "curve" stops.

(e) Check your answers by sketching each of the above curves.

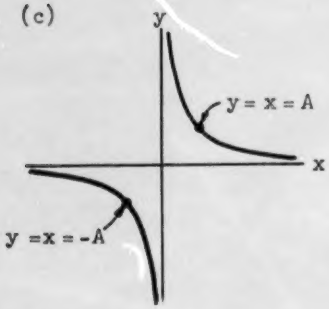
(a)



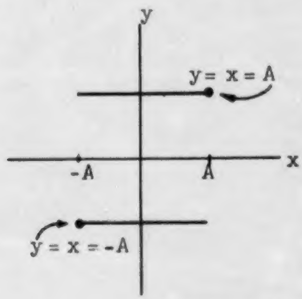
(b)



(c)

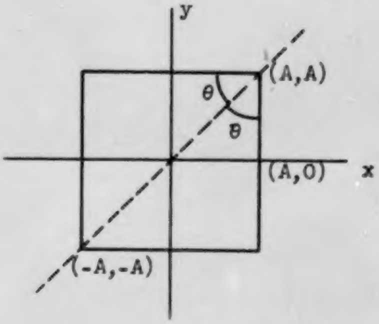


(d)



(f) Generalize the results of parts a - d.

Since interchanging y with x in each of the original equations does not change the value of the equation, all of the relations are symmetric about the line $y = x$. If the curve is smooth as in a - c, the line $y = x$ intersects the curve at an angle of 90° and the slope is -1. In d, one cannot see this generalization until we graph all parts of the relation which graphs as a square. From the figure, one may see that the line $y = x$ bisects the angle made by the two parts of the graph at the point where $y = x$. In general, for symmetric relations (with respect to y and x), if the curve is smooth at $y = x$, it has a slope of -1 at these points ($\perp$ to $y = x$). If the curve is not smooth, each branch makes equal angles with the line $y = x$.



Teachers' Commentary

Chapter 3

LIMITS AND CONTINUITY

The analytical definition of continuity formulated by Cauchy in 1821 helped to provide a sound mathematical basis for calculus. The definition may be best understood and explained in terms of the notion of limit of a function, a concept which is at the heart of differential and integral calculus and serves as a tool for considerations throughout the course.

This chapter is important in that it treats the ideas most basic to the calculus. It includes an epsilon-delta treatment of limits in order to express the ideas analytically.

In Chapters 1 and 2, efforts were made to establish a clear, intuitive notion of the limit concept. If these efforts were successful then the $\epsilon - \delta$ definition in this chapter should not appear to be a strange innovation. Instead, the student should view it as a convenient formulization of the ideas previously encountered. Definition 3-2 is not intended to replace reasoning based on correct intutive ideas of limit, but rather to provide a basis for proof in precise terms. We believe that before any argument is presented in $\epsilon - \delta$ form it should be reasoned informally by the student. Exercises have been thoughtfully selected to pave the way for the $\epsilon - \delta$ definition. Techniques involving the use of ϵ and δ are rationalized; the student is provided with some basis for locating a δ which will work. You should note that we do not try to find the "best" δ (which would entail severe difficulties even in the simplest examples); instead we are content to look for any δ which will work.

The ideas met in Chapter 3 are admittedly difficult. The student's initial exposure to these ideas may not appear to infect him. Encourage him to reread the chapter from time to time throughout the year; mastery will result from repeated exposure.

Ideas about limits are fundamental to the calculus, but the real meat of the course begins in Chapter 4.

Solutions Exercises 3-1

1. Consider $f : x \longrightarrow [x] + [-x]$. (See Section A2-1 for discussion of $[x]$.)

For $x = n$, where n is an integer, $f(x) = n + (-n) = 0$. For x not an integer,

$$n < x < n + 1 \quad \text{and} \quad -n > -x > -(n + 1)$$

whence

$$[x] = n \quad \text{and} \quad [-x] = -(n + 1) .$$

In this case, $f(x) = n + (-n - 1) = -1$.

(a) What number, if any, do the values of f approximate when x is close to 1 ? When x is close to 2 ?

$\lim\limits_{x \sim 1} f(x) = -1$

$\lim\limits_{x \sim 2} f(x) = -1$

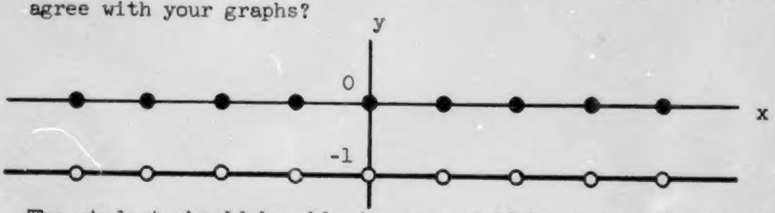
(b) What can you say of the limit of $f(x)$ as x approaches n , when n is any integer?

$\lim\limits_{x \sim n} f(x) = -1$, n an integer .

(c) Evaluate $f(n)$, where n is an integer.

$f(n) = 0$, n an integer.

(d) Sketch the graph of f . Check your answers to (a) - (c). Do they agree with your graphs?



The student should be able to say yes if he agrees with the above.

2. **For each of the following functions sketch the graph and, if possible, find the limit as x approaches 0 .**

(a) $f : x \longrightarrow \frac{x}{x}$

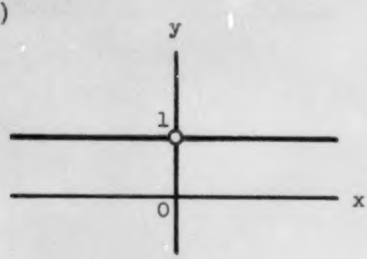
(b) $f : x \longrightarrow \frac{1}{|x|}$

(c) $f : x \longrightarrow \frac{x}{|x|}$

(d) $f : x \longrightarrow \frac{[x]}{x}$

(e) $f : x \longrightarrow \sin \frac{1}{x}$

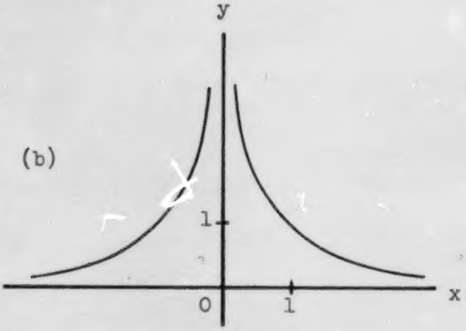
(a)



$f(x) = \frac{x}{x}$

$\lim_{x \to 0} \frac{x}{x} = 1$

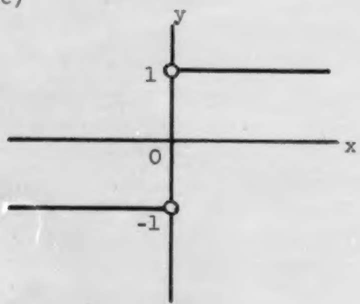
(b)



$f(x) = \frac{1}{|x|}$

$\lim_{x \to 0} \frac{1}{|x|}$ does not exist.

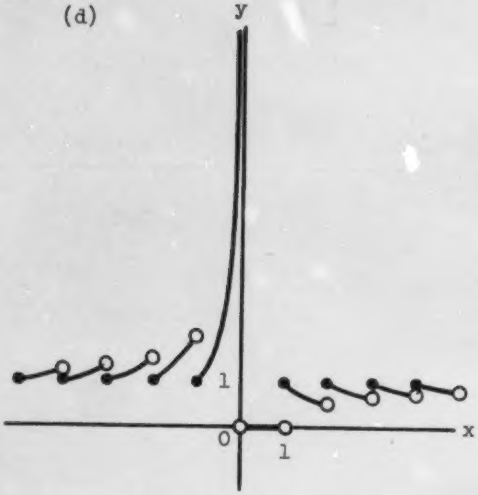
(c)



$f(x) = \frac{x}{|x|}$

$\lim_{x \to 0} \frac{x}{|x|}$ does not exist.

(d)



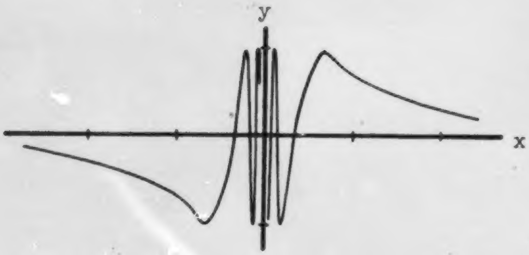
$f(x) = \frac{[x]}{x}$

$\lim_{x \to 0} \frac{[x]}{x}$ does not exist.

Interval	$[x]$	$f(x) = \frac{[x]}{x}$
(0,1)	0	0
(1,2)	1	$\frac{1}{x}$
(2,3)	2	$\frac{2}{x}$
(3,4)	3	$\frac{3}{x}$
(-1,0)	-1	$-\frac{1}{x}$
(-2,-1)	-2	$-\frac{2}{x}$
(-3,-2)	-3	$-\frac{3}{x}$

Observe that as x approaches 0 through negative values $f(x) = \frac{-1}{x} = \frac{1}{|x|}$ becomes large without bound; as x approaches 0 through positive values $f(x)$ approaches 0 (although $f(x) = 0$ for $0 < x < 1$).

(e)



$f(x) = \sin \frac{1}{x}$

$\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

TC3-2. Definition of Limit of a Function.

In some texts, the idea of limit often is expressed in words like these: "If, as x gets closer and closer to a, the values of $f(x)$ tend to the value L, then we call L the limit of $f(x)$ as x approaches a." The difficulty with this formulation, apart from the vagueness of the words, "gets closer and closer to," "tend to," is that it suggests the false notion that if x_2 is closer to a than is x_1, then $f(x_2)$ is closer to L than is $f(x_1)$.

Example TC3-2. Consider

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 . \end{cases}$$

We have $\lim\limits_{x \to 0} f(x) = 0$. Let $x_1 = \frac{1}{2n\pi}$ and $x_2 = \frac{2}{\pi(1 + 4n)}$ (n, a non-zero integer). Then $|x_2| < |x_1|$ but $|f(x_2)| > |f(x_1)|$ since $f(x_2) = \frac{2}{\pi(1 + 4n)}$ and $f(x_1) = 0$. (See Figure TC3-5a for graph of f.)

The above description of limit gives no clear idea of just how to verify that L is the limit of f as x approaches a in any particular case. We are compelled to give a definition which yields a clear-cut method of verification.

Quite early in our discussion we refer to Appendix 1-4 for an explanation of open and closed intervals. These ideas are essential to the material in this and succeeding sections. In the exercises, substantial use is made of ideas relating to the order properties of real numbers and absolute value: the student is expected to apply basic inequality theorems. The objective is to develop computational facility with absolute value as a background for proving facts about limits. As a lead into Section 3-3 we feel that it would be informative for the student to be given some numerical values for ϵ and be required to determine a δ sufficient to control the error (see, for example, Exercises 3-2, No. 11).

Solutions Exercises 3-2

The theorems of Sections A1-2 and A1-3 provide the basis for the following arguments. In the general application of the transitive property we use the strong inequality in the conclusion since a strong inequality appears at least once in the chain of reasoning (see Section A1-2). We also make extended use of the inequalities

$$\big||a| - |b|\big| \leq |a + b| \leq |a| + |b| .$$

1. Show that if $0 < |x - a| < 1$, then $|x + 2a| < 1 + 3|a|$.

   If $a < |x - a| < 1$, then

$$\begin{aligned} |x + 2a| &= |(x - a) + 3a| \\ &\leq |x - a| + |3a| \\ &\leq |x - a| + 3|a| \\ &< 1 + 3|a| . \end{aligned}$$

2. Show that if $0 < |x - a| < 1$, then $|x^3 - a^3| < (3|a^2| + 3|a| + 1)|x - a|$.

   If $0 < |x - a| < 1$, then

$$\begin{aligned} |x^3 - a^3| &= |(x - a)(x^2 + ax + a^2)| \\ &= |x - a| \cdot \big|\big((x - a) + a\big)^2 + a\big((x - a) + a\big) + a^2\big| \\ &= |x - a| \cdot |(x - a^2) + 3a(x - a) + 3a^2| \\ &\leq |x - a| \cdot [(x - a)^2 + 3|a| \cdot |x - a| + 3a^2] \\ &< 1 + 3|a| + 3a^2 . \end{aligned}$$

3. Show that if $0 < |x - 2| < 1$, then $\frac{1}{|x - 4|} < 1$. Hint: By Exercises A1-2, Number 10, if $|x - 4| > 1$, then $\frac{1}{|x - 4|} < 1$.

   We have $|x - 4| = |(x - 2) - 2|$ whence

$$|-2| - |x - 2| \leq |x - 4| \leq |x - 2| + |-2| .$$

Thus, if $0 < |x - 2| < 1$,

$$2 - 1 < |x - 4| < 1 + 2$$

or

$$1 < |x - 4| < 3$$

and

$$\frac{1}{|x - 4|} < 1 .$$

4. Show that if $|x - a| < \frac{|a|}{2}$, then $\frac{1}{x^2} < \frac{4}{a^2}$.

We have $|x| = |(x - a) + a|$, so that

$$|a| - |x - a| \leq |x| \leq |x - a| + |a| .$$

Thus, if $|x - a| < \frac{|a|}{2}$,

$$|a| - \frac{|a|}{2} < |x| < \frac{|a|}{2} + |a|$$

or

$$\frac{|a|}{2} < |x| < \frac{3|a|}{2}$$

whence

$$\frac{a^2}{4} < x^2 < \frac{9a^2}{4} ,$$

from which the result follows.

5. Show that if $0 < |x - 1| < 1$, then $|4x + 1| < 9$ and $\left|\frac{1}{x + 2}\right| < 1$.

If $0 < |x - 1| < 1$, we have

$$\begin{aligned} |4x + 1| &= |4(x - 1) + 5| \\ &\leq 4|x - 1| + 5 \\ &< 4 \cdot 1 + 5 \\ &\leq 9 . \end{aligned}$$

Also, if $0 < |x - 1| < 1$, we have

$$\begin{aligned} |x + 2| &= |(x - 1) + 3| \\ &\geq 3 - |x - 1| \\ &> 3 - 1 \\ &\geq 2 \end{aligned}$$

whence

$$\left|\frac{1}{x + 2}\right| < \frac{1}{2} < 1 .$$

6. Show that if $0 < |x - 2| < 1$, then $|x + 1| < 4$ and $\left|\frac{1}{x^2 + 2x + 4}\right| < 1$.

If $|x - 2| < 1$, then

$$\begin{aligned} |x + 1| &= |(x - 2) + 3| \\ &\le |x - 2| + 3 \\ &< 4 . \end{aligned}$$

Since $x^2 + 2x + 4 = \big((x - 2) + 2\big)^2 + 2\big((x - 2) + 2\big) + 4$

$$\begin{aligned} &= (x - 2)^2 + 6(x - 2) + 12 \\ |x^2 + 2x + 4| &\ge 12 - |(x - 2)^2 + 6(x - 2)| \\ &\ge 12 - [(x - 2)^2 + 6|x - 2|] . \end{aligned}$$

Thus, if $0 < |x - 2| < 1$,

$$\begin{aligned} |x^2 + 2x + 4| &> 12 - (1 + 6) \\ &\ge 5 . \end{aligned}$$

Finally, if $0 < |x - 2| < 1$, we have

$$\frac{1}{|x^2 + 2x + 4|} < \frac{1}{5} < 1 .$$

7. Estimate how large $x^2 + 1$ can become if x is restricted to the open interval $-3 < x < 1$.

If $-3 < x < 1$ then $3 > -x > -1$ whence

$$|x| < 3$$

and $$x^2 < 9 ,$$

so that $$x^2 + 1 < 10 .$$

8. Use inequality properties to find a positive number M such that $0 < |x - 1| < 3$ for all x and
(a) $|x^2 + 2x + 4| \le M$
(b) $|3x^2 - 2x + 3| \le M$.

We are required to submit any positive number M satisfying the given inequalities. It is not necessary to find the smallest possible number M.

The problem is included here to give the student preparatory experiences for Section 3-3. Because of this, the strategy is more valuable to the student than the actual solution.

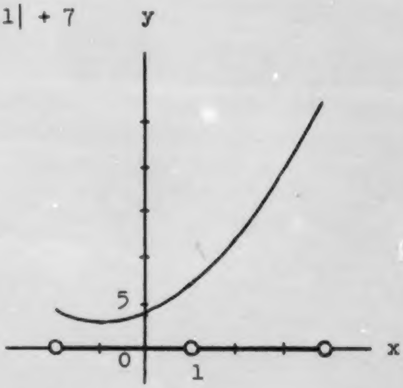
(a) $|x^2 + 2x + 4| \le M$.

For $0 < |x - 1| < 3$,

$$\begin{aligned} |x^2 + 2x + 4| &= \left|\big((x - 1) + 1\big)^2 + 2\big((x - 1) + 1\big) + 4\right| \\ &= |(x - 1)^2 + 4(x - 1) + 7| \\ &\le (x - 1)^2 + 4|x - 1| + 7 \\ &< 3^2 + 4 \cdot 3 + 7 \\ &\le 28 . \end{aligned}$$

We may take M as any number, $M \ge 28$.

The graph $y = |x^2 + 2x + 4|$ shows that any number $M \ge 28$ will serve.



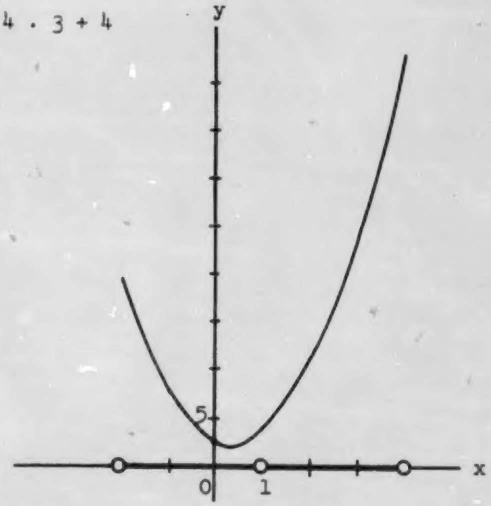
(b) $|3x^2 - 2x + 3| \le M$.

If $0 < |x - 1| < 3$, then

$$\begin{aligned} |3x^2 - 2x + 3| &= \left|3\big((x - 1) + 1\big)^2 - 2\big((x - 1) + 1\big) + 3\right| \\ &= |3(x - 1)^2 + 4(x - 1) + 4| \\ &\le 3(x - 1)^2 + 4|x - 1| + 4 \\ &< 3 \cdot 3^2 + 4 \cdot 3 + 4 \\ &\le 43 . \end{aligned}$$

We take $M \ge 43$.

The graph of $y = |3x^2 - 2x + 3|$ shows that any number $M \ge 43$ will do.



9. (a) **Show that if $0 < |x - 3| < 1$ and $0 < |x - 3| < \frac{\epsilon}{7}$, then $|x^2 - 9| < \epsilon$.**

$$|x^2 - 9| = |(x - 3)\big((x - 3) + 6\big)|$$
$$\leq |x - 3| \cdot (|x - 3| + 6) .$$

Thus, if $0 < |x - 3| < 1$ and $0 < |x - 3| < \frac{\epsilon}{7}$,

$$|x^2 - 9| < \frac{\epsilon}{7}(1 + 6)$$

or

$$|x^2 - 9| < \epsilon .$$

(b) Show that the pair of inequalities $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{7}$ (or $\delta \leq \min(1, \frac{\epsilon}{7})$) is satisfied by $\delta = \frac{\epsilon}{7 + \epsilon}$.

For $\epsilon > 0$,

$$\frac{\epsilon}{7 + \epsilon} = \frac{(7 + \epsilon) - 7}{7 + \epsilon} = \frac{7 + \epsilon}{7 + \epsilon} - \frac{7}{7 + \epsilon}$$
$$= 1 - \frac{7}{7 + \epsilon}$$
$$< 1 \quad \left(\text{since } \frac{7}{7 + \epsilon} > 0 \right).$$

Also, for $\epsilon > 0$,

$$\frac{\epsilon}{7 + \epsilon} = \epsilon \cdot \frac{1}{7 + \epsilon} < \epsilon \cdot \frac{1}{7} \quad \left(\text{since } \frac{1}{7 + \epsilon} < \frac{1}{7} \right).$$

Since $\frac{\epsilon}{7 + \epsilon} < \min(1, \frac{\epsilon}{7})$, the result follows.

10. Find a number $M \geq 1$ such that $\left|\frac{x + 4}{x - 4}\right| \leq M$ for all x such that $0 < |x - 2| < 1$. (See No. 3 above.)

If $0 < |x - 2| < 1$ then $\frac{1}{|x - 4|} < 1$ from Number 3 and

$$|x + 4| = |(x - 2) + 6|$$
$$\leq |x - 2| + 6$$
$$< 1 + 6 \leq 7 .$$

Thus, under these conditions,

$$\left|\frac{x + 4}{x - 4}\right| < |x + 4| < 7 .$$

We take M as any number, $M \geq 7$.

11. For the given value of ϵ, find a number δ such that if $0 < |x - 3| < \delta$, $|x^2 - 9| < \epsilon$.

(a) $\epsilon = 0.1$

(b) $\epsilon = 0.01$

Is your choice of δ in (b) acceptable as an answer in (a)? Explain.

$$\begin{aligned} |x^2 - 9| &= |x - 3| \cdot |(x - 3) + 6| \\ &\leq |x - 3| \cdot (|x - 3| + 6) \\ &< \delta(\delta + 6) . \end{aligned}$$

(At the last line we used $0 < |x - 3| < \delta$.) For convenience, we restrict δ so that $\delta \leq 1$. Then, under this condition, $|x^2 - 9| < 7\delta$.

(a) To insure that $|x^2 - 9| < 0.1$ we may take $\delta = \frac{0.1}{7} = \frac{1}{70}$.

(b) To insure that $|x^2 - 9| < 0.01$ we take $\delta = \frac{0.01}{7} = \frac{1}{700}$.

The choice, $\delta = \frac{1}{700}$, is acceptable in (a), for if $0 < |x - 3| < \frac{1}{700}$ then

$$|x^2 - 9| < 7\delta \leq 0.01 < 0.1 .$$

12. For the following functions, find the limit L as x approaches a . For each value of ϵ , exhibit a number δ such that $|f(x) - L| < \epsilon$ whenever $|x - a| < \delta$.

(a) $f(x) = 3x - 2$, $a = \frac{1}{2}$.

(b) $f(x) = mx + b$, $(m \neq 0)$.

(c) $f(x) = 1 + x^2$, $a = 0$.

(a) $\lim\limits_{x \to \frac{1}{2}} (3x - 2) = -\frac{1}{2}$.

We have

$$\begin{aligned} |(3x - 2) - (-\tfrac{1}{2})| &= |3x - \tfrac{3}{2}| \\ &= 3|x - \tfrac{1}{2}| . \end{aligned}$$

We wish to find a δ such that whenever $|x - \frac{1}{2}| < \delta$ then

$$|(3x - 2) = (-\tfrac{1}{2})| < \epsilon .$$

We take $\delta = \frac{\epsilon}{3}$. Then if $|x - \frac{1}{2}| < \delta$,

$$\begin{aligned} |(3x - \tfrac{1}{2}) - \tfrac{1}{2}| &= 3|x - \tfrac{1}{2}| \\ &< 3\delta \\ &\leq \epsilon . \end{aligned}$$

(b) $f(x) = mx + b$, $(m \neq 0)$.

$$\lim_{x \to a} f(x) = ma + b$$

$$\begin{aligned} |(mx + b) - (ma + b)| &= |m(x - a)| \\ &= |m| \cdot |x - a| . \end{aligned}$$

We wish to find a δ such that whenever $|x - a| < \delta$ then

$$|m| \cdot |x - a| < \epsilon .$$

We take $\delta = \frac{\epsilon}{|m|}$. For this choice of δ , whenever $|x - a| < \delta$,

$$\begin{aligned} |(mx + b) - (ma + b)| &= |m| \cdot |x - a| \\ &< |m| \cdot \delta \\ &\leq \epsilon . \end{aligned}$$

(c) $f(x) = 1 + x^2$, $a = 0$.

$$\lim_{x \to 0} (1 + x^2) = 1$$

$$|(1 + x^2) - 1| = x^2 .$$

We wish to find a $\delta > 0$ such that $|(1 + x^2) - 1| < \epsilon$ whenever $|x - 0| < \delta$. We take $\delta = \sqrt{\epsilon}$. For this choice of δ , if $|x - 0| < \delta$,

$$\begin{aligned} |(1 + x^2) - 1| &= x^2 \\ &< \delta^2 \\ &\leq \epsilon . \end{aligned}$$

TC3-3. Epsilonic Technique.

The importance of technical mastery is lost on some students, usually among the brightest. It may be necessary to emphasize for them the connection between mechanical skills and a conceptual grasp of the subject. Just as an accomplished musician can perceive the essence of a composition without stumbling over individual notes, the accomplished user of mathematics must have enough mechanical facility to be above distraction by mechanical details. The same remarks apply to the technical sections of Chapters 4 to 10.

A great deal of pedagogical consideration has gone into the composition of Section 3-3. The student should develop an operationally satisfactory way of working with the idea of limit. Memorization of Definition 3-2 is (certainly) not sufficient. Nevertheless, definitions are like the fixed stars. They give the student a firm criterion for knowing where he is.

We wish to cultivate the attitude of inquiry in which the student asks himself the following questions:

1. Do I have suitable approximations for L ? (The answer should be easy since the approximations are usually taken at endpoints or at interior points of a defined interval.)
2. Do I have a candidate for L ? If so, what is it?
3. How shall I test the candidate to see if it is the limit? Can I keep the error within any given tolerance ϵ by confining the points x to a suitable δ-neighborhood of a ?

It is easy to show that if, for an arbitrary $\epsilon > 0$, there exists a control, $\delta > 0$, then any smaller positive number δ^* will certainly suffice for the same ϵ . For, let there exist a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$. Further, let δ^* be any number, $0 < \delta^* < \delta$. It follows at once that for all x such that $0 < |x - a| < \delta^*$ we have $0 < |x - a| < \delta$; whence, for all these x , $|f(x) - L| < \epsilon$.

We are not concerned with the largest δ that gives the desired degree of control over the error tolerance ϵ ; rather, we seek any number δ which is sufficient. The task is often simplified if we agree to restrict δ to numbers less than 1 . This restriction simply means that we are focusing our attention on the deleted interval $\{x : 0 < |x - a| < \delta < 1\}$.

We noted in Section 3-3 (Step 1. Simplification) that frequently we are able to find a simple function $g : \delta \longrightarrow c\delta$, c a positive constant. This is the case because we are dealing primarily with functions which have continuous derivatives on a full neighborhood of a. From the fact that f is continuous, we have

$$L = \lim_{x \to a} f(x) = f(a) .$$

From the fact that f' is continuous on a neighborhood we know that $|f'(x)|$ is also continuous on any closed interval centered at a within the neighborhood (composition of continuous functions Section 3-6). Consequently, $|f'(x)|$ has a maximum value on the interval. (Extreme Value Theorem, Section 3-7). Let K be any value greater than the maximum, so that $|f'(x)| < K$ on the interval. Now, assuming $0 < |x - a| < \delta$, where this deleted δ-neighborhood lies within the interval there exists a value ξ within the δ-neighborhood (Law of the Mean, Chapter 5) such that

$$\begin{aligned} |f(x) - L| &= |f(x) - f(a)| \\ &= |f'(\xi)(x - a)| \\ &= |f'(\xi)| \cdot |x - a| \\ &< K|x - a| \\ &< K\delta . \end{aligned}$$

For a brief explanation of "upper bound" and "lower bound" we refer you to Exercises A1-4, No. 4.

The method of bounding the denominator in Example 3-3e is given because it is a routine procedure conforming to the letter of our general outline. A short cut is to anticipate the problem of bounding x away from 0 at line (1). We may recognize at once that, since $|x - a| < \delta$, the distance from x to a is no larger than δ; we may then keep x away from 0 by requiring δ to be less than the distance $|a|$ of a from 0. To achieve this we may take $\delta \leq \frac{|a|}{2}$.

For your reference we list the following generalities:

1. The definition of limit employs only values of x different from a.
2. Limit is a local property (sometimes called a property in the small) involving the behavior of a function within any (deleted) neighborhood of a point.

3. The existence of the limit of f at a point implies that f is defined for some values of x in every deleted neighborhood of a ; that is, f(x) exists for some values of x arbitrarily near a .

4. The limit is independent of the choice of the deleted neighborhood of a .

5. The assertion that the function f has the limit L as x approaches a is not the same as saying $f(a) = L$, nor is it the same as saying that L is an upper (lower) bound of f(x) .

6. The value of δ depends upon the value of ϵ (exception: $f(x) = c$, c constant).

A careful distinction should be made between the analysis of a problem and its exposition. This is particularly necessary in the case of limit proofs. Steps 1 and 2 show how the solution is found: in Step 1 we examine the problem and set up a simplified model; in Step 2 we outline our plan of attack or strategy. Step 3 is the actual proof where it is verified that the solution has been found. An attempt should be made to develop elegance of style in presenting proofs.

Solutions Exercises 3-3

In the following epsilonic arguments the analysis (Steps 1 and 2 in the pattern of the text) precedes the proof. We make liberal use of the inequalities

$$\big||a| - |b|\big| \leq |a + b| \leq |a| + |b|$$

(Section A1-3). In the selection of δ (in Numbers 4b - 4g) it is expedient to restrict δ by the auxiliary conditions that $\delta \leq 1$. The proof (verification) is simplified by an application of Exercises A1-3a, Number 5b.

1. Prove $\lim_{x \to 4} (\frac{1}{2} x - 3) = -1$: obtain an upper bound $g(\delta)$ for the absolute error and find δ in terms of ϵ .

In this problem we write out the steps in detail.

To prove that $\lim_{x \to 4} (\frac{1}{2} x - 3) = -1$.

For each $\epsilon > 0$ obtain a δ .

Show: if $0 < |x - 4| < \delta$, then $|(\frac{1}{2} x - 3) - (-1)| < \epsilon$.

Step 1.

(a)
$$|(\tfrac{1}{2} x - 3) - (-1)| = |\tfrac{1}{2} x - 2| = \tfrac{1}{2} |x - 4| .$$

(b) If $0 < |x - 4| < \delta$,

$$|(\tfrac{1}{2}x - 3) - (-1)| = \tfrac{1}{2}|x - 4|$$
$$< \tfrac{1}{2}\delta .$$

(c) Take $g(\delta) = \frac{1}{2}\delta$.

Step 2. To make $g(\delta) \leq \epsilon$, set $\delta = 2\epsilon$.

Step 3. If $\delta = 2\epsilon$ and $0 < |x - 4| < \delta$, then

$$|(\tfrac{1}{2}x - 3) - (-1)| = \tfrac{1}{2}|x - 4|$$
$$< \tfrac{1}{2}(2\epsilon)$$
$$\leq \epsilon .$$

2. Give arguments that prove

(a) $\lim_{x\to a} c = c$, c any constant.

(b) $\lim_{x\to a} x = a$.

(c) $\lim_{x\to a} kx = ka$, k any constant.

(Use the results of Example 3-3a of the text for parts b and c.)

(a) $\lim_{x\to a} c = c$, c constant.

Statement of problem:

For each $\epsilon > 0$ we obtain a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|c - c| < \epsilon$.

Since $|c - c| = 0$ is less than $\epsilon > 0$ for all δ we arbitrarily take any positive number for δ , say $\delta = 1$.

Then for $\delta = 1$, whenever $0 < |x - a| < \delta$, we have $|c - c| < \epsilon$.

(b) $\lim_{x\to a} x = a$.

From Example 3-3a we have

$$\lim_{x\to a}(mx + b) = ma + b \text{ , } m \neq 0 \text{ ,}$$

whence for $m = 1$, $b = 0$,

$$\lim_{x\to a} x = a .$$

(c) $\lim_{x \to a} kx = ka$, k constant.

If $k \neq 0$, the result is a direct consequence of Example 3-3a; if $k = 0$, the result follows from part (a).

3. Invoke the definition directly to prove the existence of the limits to Problem 2.

(a) See answer to Number 2(a).

(b) $\lim_{x \to a} x = a$.

We follow the pattern of Example 3-3a in Step 1. Then take $g(\delta) = \delta$.

To make $g(\delta) \leq \epsilon$ we take $\delta = \epsilon$.

Thus, if $\delta = \epsilon$ and $0 < |x - a| < \delta$, then $|x - a| < \delta \leq \epsilon$.

(c) $\lim_{x \to a} kx = ka$, k constant.

We follow the pattern of Example 3-3a through Step 2 and take $\delta = \frac{\epsilon}{|m|}$ where $m = k$.

For $\delta = \frac{\epsilon}{|k|}$ and $0 < |x - a| < \delta$,

$$\begin{aligned} |kx - ka| &= |k| \cdot |x - a| \\ &< |k|\delta \\ &\leq \epsilon . \end{aligned}$$

4. In each of the following guess the limit, and then prove that your guess is correct: give an expression for $g(\delta)$ and find δ in terms of ϵ .

(a) $\lim_{x \to 0} \frac{1}{1 + x^2}$

(b) $\lim_{x \to 3} \frac{x^2(x - 3)}{x - 3}$

(c) $\lim_{x \to a} \frac{x^3 - a^3}{x - a}$

(d) $\lim_{x \to 1} \frac{x + 1}{x^2 + 1}$

(e) $\lim_{x \to 2} \frac{x^2 - 4}{x^3 - 8}$

(f) $\lim_{x \to 0} \frac{x^3 - 3x - 1}{x + 2}$

(g) $\lim_{x \to 1} \frac{4x^2 - 3x - 1}{x + 2}$

We omit repetitious material. The statement of the problem follows the pattern in the text.

3-3

(a) $\lim_{x \to 0} \frac{1}{1 + x^2} = 1$.

$$\left|\frac{1}{1 + x^2} - 1\right| = \left|\frac{x^2}{1 + x^2}\right| = \frac{x^2}{1 + x^2}$$

$$< x^2 \qquad \text{(since } 1 + x^2 \geq 0 \text{)}$$

$$< \delta^2 \qquad \text{if } 0 < |x| < \delta \text{ .}$$

Take $g(\delta) = \delta^2$ and set $\delta = \sqrt{\epsilon}$.

Verification:

If $\delta = \sqrt{\epsilon}$ and $0 < |x| < \delta$, then $\left|\frac{1}{1 + x^2} - 1\right| < \delta^2 \leq \epsilon$.

(b) $\lim_{x \to 3} \frac{x^2(x - 3)}{x - 3} = 9$.

$$\left|\frac{x^2(x - 3)}{x - 3} - 9\right| = |x^2 - 9| \qquad \text{for } x \neq 3$$

$$= |x - 3| \cdot |(x - 3) + 6|$$

$$\leq |x - 3| \cdot (|x - 3| + 6)$$

$$< \delta(\delta + 6) \text{ .}$$

(At the last line we used $0 < |x - 3| < \delta$.)

For convenience we restrict δ by requiring that $\delta \leq 1$. Under this condition if $0 < |x - 3| < \delta$

$$\left|\frac{x^2(x - 3)}{x - 3} - 9\right| < 7\delta \text{ .}$$

Take $g(\delta) = 7\delta$. Obtain a δ satisfying the two conditions $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{7}$, simultaneously. One way to do this is to take $\delta = \frac{\epsilon}{7 + \epsilon}$ (Exercises 3-2, No. 9b).

Verification: If $\delta = \frac{\epsilon}{7 + \epsilon}$ and $0 < |x - 3| < \delta$

$$\left|\frac{x^2(x - 3)}{x - 3} - 9\right| < 7\delta$$

$$\leq 7 \cdot \frac{\epsilon}{7 + \epsilon}$$

$$\leq \frac{7}{7 + \epsilon} \cdot \epsilon$$

$$< \epsilon \qquad \text{(since } \frac{7}{7 + \epsilon} < 1 \text{).}$$

(c) $\lim_{x\ a} \dfrac{x^3 - a^3}{x - a} = 3a^2$.

We have $\dfrac{x^3 - a^3}{x - a} = x^2 + ax + a^2$ for $x \neq a$, whence

$$\begin{aligned}
\left|\frac{x^3 - a^3}{x - a} - 3a^2\right| &= |x^2 + ax - 2a^2| \\
&= \left|\big((x - a) + a\big)^2 + a\big((x - a) + a\big) - 2a^2\right| \\
&= |(x - a)^2 + 3a(x - a)| \\
&= |x - a| \cdot |(x - a) + 3a| \\
&\leq |x - a| \cdot (|x - a| + 3|a|) \\
&< \delta(\delta + 3|a|)
\end{aligned}$$

(At the last line we used $0 < |x - a| < \delta$.)

For convenience we restrict δ by requiring $\delta \leq 1$. Under this condition if $0 < |x - a| < \delta$,

$$\left|\frac{x^3 - a^3}{x - a} - 3a^2\right| < \delta(1 + 3|a|) .$$

We choose δ to satisfy the conditions $\delta \leq 1$ and $\delta \leq \dfrac{\epsilon}{1 + 3|a|}$; i.e., $\delta \leq \min\{1, \dfrac{\epsilon}{1 + 3|a|}\}$. A convenient way to satisfy these conditions is to take $\delta = \dfrac{\epsilon}{1 + 3|a| + \epsilon}$ (Exercises A1-3, No. 6b).

Verification:

If $\delta = \dfrac{\epsilon}{1 + 3|a| + \epsilon}$ and $0 < |x - a| < \delta$, then

$$\begin{aligned}
\left|\frac{x^3 - a^3}{x - a} - 3a^2\right| &= |x - a| \cdot |(x - a) + 3a| \\
&\leq |x - a| \cdot (|x - a| + 3|a|) \\
&< \delta(1 + 3|a|) \\
&\leq \frac{\epsilon}{1 + 3|a| + \epsilon} \cdot (1 + 3|a|) \\
&\leq \epsilon \cdot \frac{1 + 3|a|}{1 + 3|a| + \epsilon} \\
&< \epsilon \qquad \left(\text{since } \frac{1 + 3|a|}{1 + 3|a| + \epsilon} < 1\right) .
\end{aligned}$$

(d) $\lim\limits_{x \to 1} \frac{x + 1}{x^2 + 1} = 1$.

$$\left|\frac{x + 1}{x^2 + 1} - 1\right| = \left|\frac{x - x^2}{x^2 + 1}\right| = \frac{|x - 1| \cdot |x|}{1 + x^2}$$

$$\le |x - 1| \cdot |x| \qquad (\text{since } 1 + x^2 \ge 1)$$

$$\le |x - 1| \cdot |(x - 1) + 1|$$

$$\le |x - 1| \cdot (|x - 1| + 1)$$

$$< \delta(\delta + 1) \qquad \text{if } |x - 1| < \delta .$$

For convenience we restrict δ by requiring $\delta \le 1$. Under this condition if $|x - 1| < \delta$

$$\left|\frac{x + 1}{x^2 + 1} - 1\right| < \delta(\delta + 1)$$

$$\le 2\delta .$$

To satisfy the two conditions $\delta \le 1$ and $\delta \le \frac{\epsilon}{2}$ simultaneously we take (for convenience) $\delta = \frac{\epsilon}{2 + \epsilon}$.

Verification:

If $\delta = \frac{\epsilon}{2 + \epsilon}$ and $|x - 1| < \delta$,

$$\left|\frac{x + 1}{x^2 + 1} - 1\right| < 2\delta$$

$$\le 2 \cdot \frac{\epsilon}{2 + \epsilon}$$

$$\le \frac{2}{2 + \epsilon} \cdot \epsilon$$

$$< \epsilon \qquad (\text{since } \frac{2}{2 + \epsilon} < 1) .$$

(e) $\lim\limits_{x \to 2} \frac{x^2 - 4}{x^3 - 8} = \frac{1}{3}$.

$$\left|\frac{x^2 - 4}{x^3 - 8} - \frac{1}{3}\right| = \left|\frac{x + 2}{x^2 + 2x + 4} - \frac{1}{3}\right| \qquad (x \ne 2)$$

$$= \left|\frac{-x^2 + x + 2}{3(x^2 + 2x + 4)}\right|$$

$$= \frac{1}{3}\left|\frac{(x - 2)(x + 1)}{x^2 + 2x + 4}\right| .$$

For convenience we restrict δ by requiring $\delta \leq 1$. Under this condition, if $0 < |x - 2| < \delta$,

$$\left|\frac{x^2 - 4}{x^3 - 8} - \frac{1}{3}\right| < \frac{1}{3}\,|(x - 2)(x + 1)| \quad \text{(from Exercises 3-2, No. 6)}$$

$$\leq \frac{1}{3}\,|x - 2| \cdot |(x - 2) + 3|$$

$$\leq \frac{1}{3}\,|x - 2| \cdot (|x - 2| + 3)$$

$$< \frac{1}{3}\,\delta(1 + 3)$$

$$\leq \frac{4}{3}\,\delta\,.$$

We wish to obtain a value δ satisfying the two conditions $\delta \leq 1$ and $\delta \leq \frac{3}{4}\epsilon$, simultaneously. One way to satisfy the conditions is to set

$$\delta = \frac{\frac{3}{4}\epsilon}{1 + \frac{3}{4}\epsilon} = \frac{3\epsilon}{4 + 3\epsilon} \quad \text{(Exercises A1-3, No. 6b).}$$

Verification:

If $\delta = \frac{3\epsilon}{4 + 3\epsilon}$ and $0 < |x - 2| < \delta$,

$$\left|\frac{x^2 - 4}{x^3 - 8} - \frac{1}{3}\right| \leq \frac{1}{3}\,|x - 2| \cdot (|x - 2| + 3)$$

$$< \frac{4}{3}\,\delta$$

$$\leq \frac{4}{3} \cdot \frac{3\epsilon}{4 + 3\epsilon}$$

$$\leq \frac{12}{3(4 + 3\epsilon)} \cdot \epsilon$$

$$< \epsilon \qquad \left(\text{since } \frac{12}{3(4 + 3\epsilon)} < 1\right).$$

(f) $\lim\limits_{x \to 0} \dfrac{x^3 - 3x - 1}{x + 2} = -\dfrac{1}{2}$.

$$\left|\frac{x^3 - 3x - 1}{x + 2} - \left(-\frac{1}{2}\right)\right| = \left|\frac{2x^3 - 5x}{2(x + 2)}\right|\,.$$

For convenience we rectrict δ by requiring $\delta \leq 1$. Thus if $0 < |x| < \delta \leq 1$, $|x + 2| \geq 2 - |x| \geq 1$ and $\frac{1}{|x + 2|} \leq 1$.

Thus if $\delta \leq 1$ and $0 < |x| < \delta$, we have

$$\left|\frac{x^3 - 3x - 1}{x + 2} - (-\tfrac{1}{2})\right| = \left|\frac{2x^3 - 5x}{2(x + 2)}\right|$$

$$\leq \frac{1}{2}\,|2x^3 - 5x|$$

$$\leq \frac{1}{2} \cdot |x| \cdot |2x^2 - 5|$$

$$\leq |x| \cdot (x^2 + \tfrac{5}{2})$$

$$< \delta \cdot (1 + \tfrac{5}{2})$$

$$< 4\delta \qquad \text{(since } \tfrac{7}{2}\delta < 4\delta\text{)}.$$

To satisfy the two conditions $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{4}$ simultaneously, we take, for convenience, $\delta = \frac{\epsilon}{4 + \epsilon}$.

Verification:

Set $\delta = \frac{\epsilon}{4 + \epsilon}$ in the statement of the problem, as in (a) - (e). In the last step we have, under the conditions that $0 < |x - 0| < \delta$,

$$\left|\frac{x^3 - 3x - 1}{x + 2} - (-\tfrac{1}{2})\right| < 4\delta$$

$$\leq 4 \cdot \frac{\epsilon}{4 + \epsilon}$$

$$< \epsilon \qquad \text{(since } \frac{4}{4 + \epsilon} < 1\text{)}.$$

(g) $\lim_{x \to 1} \frac{4x^2 - 3x - 1}{x + 2} = 0$.

$$\left|\frac{4x^2 - 3x - 1}{x + 2}\right| = \frac{|(x - 1)(4x + 1)|}{|x + 2|}$$

$$= \frac{|x - 1| \cdot |4(x - 1) + 5|}{|x + 2|} .$$

We restrict δ so that $\delta \leq 1$. Under this rectriction, if $0 < |x - 1| < \delta$, we have

$$\left|\frac{4x^2 - 3x - 1}{x + 2}\right| < |x - 1| \cdot |4(x - 1) + 5| \quad \text{(Exercises 3-2, No. 5)}$$

$$\leq |x - 1| \cdot (4|x - 1| + 5)$$

$$< \delta(4\delta + 5)$$

$$\leq 9\delta .$$

We require that $\delta \le \min\{1, \frac{\epsilon}{9}\}$. This condition is satisfied if we **take** $\delta = \frac{\epsilon}{9 + \epsilon}$.

Verification:

If $\delta = \frac{\epsilon}{9 + \epsilon}$ and $0 < |x - 1| < \delta$, then

$$\left|\frac{4x^2 - 3x - 1}{x + 2}\right| < |(x - 1)(4x + 1)|$$

$$\le |x - 1| \cdot (4|x - 1| + 5)$$

$$< 9\delta$$

$$\le 9 \cdot \frac{\epsilon}{9 + \epsilon}$$

$$< \epsilon \qquad \left(\text{since } \frac{\epsilon}{9 + \epsilon} < 1\right)$$

TC3-4. Limit Theorems.

Theorems become tools to the student only in so far as he understands the hypotheses and appreciates conclusions that derive therefrom. Graphical considerations are usually helpful in visualizing abstract properties and proofs (in particular, Theorems 3-4a, b, c, f). Proofs of the theorems should be made plausible, but memorization of formal proofs is inconsistant with the philosophy of this course.

The statement $\lim_{x \to a} f(x) = L$ simply means that L satisfies the conditions of Definition 3-2. It is conceivable that another number M might satisfy the same conditions. Then the statements $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$ would both be true. If this could happen, then $\lim_{x \to a} f(x)$ could have no meaning by itself. Consequently, each limit theorem would need to be appropriately interpreted; for example, Theorem 3-4c would be stated:

Theorem 3-4c. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then $L + M$ is a limit of $f + g$ at $x = a$.

This interpretation is unnecessary, since $\lim_{x \to a} f(x)$, if it exists, is unique.

THEOREM. If L and M are limits of f at a, that is, if $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$, then $L = M$.

Proof. Suppose $L > M$, take $\epsilon = \frac{L - M}{3} > 0$, then there are $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$-\epsilon < L - f(x) < \epsilon \qquad \text{for } 0 < |x - a| < \delta_1,$$

$$-\epsilon < f(x) - M < \epsilon \qquad \text{for } 0 < |x - a| < \delta_2 .$$

Hence both inequalities hold for

$$0 < |x - a| < \min(\delta_1, \delta_2) .$$

Thus $0 < L - M < 2\epsilon$. Impossible.

Sometimes assumptions are left tacit. For example, in the proof of Theorem 3-4b, ϵ^* is, of course, subject to the conditions imposed upon ϵ, that is, $\epsilon^* > 0$.

In Theorem 3-4c, there is a tolerance, ϵ, for the sum $(f + g)$ and tolerances, ϵ_1 and ϵ_2, for the addends (f and g, respectively). It is sufficient that these tolerances satisfy the condition $\epsilon_1 + \epsilon_2 \leq \epsilon$: for convenience (and for definiteness) we take $\epsilon_1 = \epsilon_2 = \frac{\epsilon}{2}$, in the text. The control δ_1 maintains the tolerance $\epsilon_1 = \frac{\epsilon}{2}$ for f and control δ_2 maintains $\epsilon_2 = \frac{\epsilon}{2}$ for g. It follows that $\delta \leq \min(\delta_1, \delta_2)$ will maintain the tolerance $\frac{\epsilon}{2}$ for f and also for g.

We can obtain a slightly different proof of Theorem 3-4d by utilizing Lemma 3-4 in the following way. By hypothesis, corresponding to any positive ϵ_1, ϵ_2 we can find δ_1, δ_2 such that

$$|f(x) - L| < \epsilon_1 \quad \text{whenever} \quad 0 < |x - a| < \delta_1 ,$$

$$|g(x) - M| < \epsilon_2 \quad \text{whenever} \quad 0 < |x - a| < \delta_2 ,$$

and by Lemma 3-4 there is a δ^* such that

$$|g(x)| < \frac{3}{2} |M| \quad \text{whenever} \quad 0 < |x - a| < \delta^* .$$

Since $f(x)g(x) - LM = \big(f(x) - L\big)g(x) + L\big(g(x) - M\big)$, if $|f(x) - L| < \epsilon_1$, $|g(x) - M| < \epsilon_2$, and $|g(x)| < \frac{3}{2}|M|$ then

$$\begin{aligned} |f(x)g(x) - LM| &\le |f(x) - L| \cdot |g(x)| + |L| \cdot |g(x) - M| \\ &< \epsilon_1 \tfrac{3}{2}|M| + |L|\,\epsilon_2 \,. \end{aligned}$$

In order to remain within the tolerance ϵ we take $\epsilon_1 = \epsilon_2$ and

$$\epsilon_1 \tfrac{3}{2}|M| + |L|\epsilon_2 = \epsilon_1\left(\tfrac{3}{2}|M| + |L|\right) \le \epsilon \,.$$

Then to cover the case where $M = L = 0$ we take

$$\epsilon_1 = \epsilon_2 = \frac{\epsilon}{\frac{3}{2}|M| + |L| + 1} \,.$$

Let δ_1, δ_2, and δ^* be appropriate controls for this choice of ϵ_1 and ϵ_2 and take

$$\delta = \min\{\delta_1, \delta_2, \delta^*\} \,.$$

For this choice of δ and for

$$0 < |x - a| < \delta$$

we have

$$\begin{aligned} |f(x)g(x) - LM| &\le |f(x) - L| \cdot |g(x)| + |L| \cdot |g(x) - M| \\ &\le \frac{\epsilon}{\frac{3}{2}|M| + |L| + 1} \cdot \left(\tfrac{3}{2}|M| + |L|\right) \\ &< \epsilon \,. \end{aligned}$$

Sum notation (Section A3-2) may be used to express a linear combination

$$\phi(x) = \sum_{i=1}^{n} c_i f_i(x)$$

and state the corollary to Theorem 3-4c,

$$\begin{aligned} \lim_{x \to a} \sum_{i=1}^{n} c_i f_i(x) &= \sum_{i=1}^{n} c_i \lim_{x \to a} f_i(x) \\ &= \sum_{i=1}^{n} c_i L_i \,. \end{aligned}$$

Mathematical induction (Appendix 3) is required for the proof of the corollaries to Theorems 3-4c and 3-4d, respectively. If the student works through these proofs he will gain a deeper understanding of the fundamental results and an appreciation of the power of mathematical induction.

In Lemma 3-4 (and frequently in Chapter 14) we have ". . . a δ-neighborhood of a wherein $g(x)$ is closer to M than to zero." This means: for all x in the δ-neighborhood and in the domain of g, $g(x)$ is closer to M than to zero.

Squeeze Theorem (Proof). Let I be the deleted neighborhood of a where

$$h(x) \le f(x) \le g(x) .$$

For every $\epsilon > 0$, I contains a deleted δ-neighborhood of a wherein $|h(x) - M| < \epsilon$ and $|g(x) - M| < \epsilon$. Equivalently, we have

$$M - \epsilon < h(x) \quad , \quad g(x) < M + \epsilon$$

whenever $0 < |x - a| < \delta$. It follows that

$$M - \epsilon < f(x) < M + \epsilon$$

or

$$|f(x) - M| < \epsilon$$

whenever $0 < |x - a| < \delta$.

Solutions Exercises 3-4

∧ 1. Prove the corollary to Theorem 3-4c. The limit of a linear combination of functions is the same linear combination of the limits of the functions; i.e., if $\lim\limits_{x \to a} f_i(x) = L_i$, $i = 1, 2, \ldots, n$, then

$$\lim_{x \to a} \sum_{i=1}^{n} c_i f_i(x) = \sum_{i=1}^{n} \lim_{x \to a} c_i f_i(x)$$

$$= \sum_{i=1}^{n} c_i \lim_{x \to a} f_i(x)$$

$$= \sum_{i=1}^{n} c_i L_i \qquad \text{(see A3-2).}$$

Proof. We use the First Principle of Mathematical Induction (Appendix 3-1), and take for A_n the assertion

$$\lim_{x\to a}\sum_{i=1}^{n} c_i f_i(x) = \sum_{i=1}^{n} c_i L_i \ .$$

For $n = 1$ we have as assertion A_1,

$$\lim_{x\to a} c_1 f_1(x) = c_1 L_1 \ ,$$

which is true by Theorem 3-4b.

We now assume A_n true for $n = k$ and seek to prove that A_{k+1} is true. From the induction hypothesis,

$$\lim_{x\to a}\sum_{i=1}^{k} c_i f_i(x) = \sum_{i=1}^{k} c_i L_i \ .$$

Now

$$\begin{aligned}
\lim_{x\to a}\sum_{i=1}^{k+1} c_i f_i(x) &= \lim_{x\to a}\left(\sum_{i=1}^{k} c_i f_i(x) + c_{k+1}\, f_{k+1}(x)\right) \\
&= \lim_{x\to a}\sum_{i=1}^{k} c_i f_i(x) + \lim_{x\to a} c_{k+1}\, f_{k+1}(x) \qquad \text{(Theorem 3-4c)} \\
&= \sum_{i=1}^{k} c_i L_i + \lim_{x\to a} c_{k+1}\, f_{k+1}(x) \\
&= \sum_{i=1}^{k} c_i L_i + c_{k+1}\, L_{k+1} \qquad \text{(Theorem 3-4b)} \\
&= \sum_{i=1}^{k+1} c_i L_i \ ,
\end{aligned}$$

and assertion A_{k+1} is true. Therefore,

$$\lim_{x\to a}\sum_{i=1}^{n} c_i f_i(x) = \sum_{i=1}^{n} c_i L_i$$

holds for every natural number n .

Λ 2. Prove the corollary to Theorem 3-4d. For any polynomial function p ,

$$\lim_{x \to a} p(x) = p(a)$$

Proof. To establish the corollary, we use the First Principle of Mathematical Induction to prove first that

$$\lim_{x \to a} x^n = a^n \ , \qquad n \text{ any natural number.}$$

We take for A_n the assertion that $\lim_{x \to a} x^n = a^n$.

For $n = 1$, assertion A_1 is

$$\lim_{x \to a} x = a$$

which is true (Exercises 3-3, No. 2b).

We now assume A_k true and try to prove A_{k+1} true. The induction hypothesis is

$$\lim_{x \to a} x^k = a^k \ .$$

Now

$$\begin{aligned} \lim_{x \to a} x^{k+1} &= \lim_{x \to a} (x^k \cdot x) \\ &= \left(\lim_{x \to a} x^k\right)\left(\lim_{x \to a} x\right) && \text{(Theorem 3-4d)} \\ &= a^k \cdot a && (A_1 \text{ and induction hypothesis}) \\ &= a^{k+1} \ . \end{aligned}$$

Thus, A_{k+1} is true and

$$\lim_{x \to a} x^n = a^n \ , \quad n \text{ any natural number.}$$

To complete the proof, let

$$p(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n = \sum_{i=0}^{n} c_i x^i \ .$$

$$\lim_{x \to a} p(x) = \lim_{x \to a} \sum_{i=0}^{n} c_i x^i$$

$$= \sum_{i=0}^{n} \lim_{x \to a} c_i x^i \qquad \text{(Corollary to Theorem 3-4c)}$$

$$= \sum_{i=0}^{n} c_i \lim_{x \to a} x^i \qquad \text{(Theorem 3-4d)}$$

$$= \sum_{i=0}^{n} c_i a^i$$

$$= p(a) \; ,$$

as was to be shown.

3. Prove the corollaries to Lemma 3-4.

(a) Corollary 1. If $\lim_{x \to a} g(x) = M$ and $M \neq 0$, then there exists a neighborhood of a where $|\frac{3M}{2}| > |g(x)| > |\frac{M}{2}|$ for x in the domain of g.

Proof. Since g has limit M at a, there is a δ-neighborhood of a in which $g(x)$ is closer to M than to zero. We consider two cases: $M > 0$ and $M < 0$.

Case 1. If $M > 0$, we have $g(x) > 0$ by Lemma 3-4, and $\frac{3M}{2} > g(x) > \frac{M}{2} > 0$ or $|\frac{3M}{2}| > |g(x)| > |\frac{M}{2}| > 0$.

Case 2. If $M < 0$, we have $-M > 0$ and $\lim_{x \to a} \left(-g(x)\right) = -M > 0$. Thus, $-g(x) > 0$ by Lemma 3-4, and $\frac{-3M}{2} > -g(x) > \frac{-M}{2} > 0$. This inequality is equivalent to $|\frac{-3M}{2}| > |-g(x)| > |\frac{-M}{2}| > 0$ or $|\frac{3M}{2}| > |g(x)| > |\frac{M}{2}| > 0$, which is the same as the inequality obtained in Case 1.

(b) Corollary 2. A limit of a function whose values are nonnegative is nonnegative.

Proof. Let $g(x) \geq 0$ and let $\lim_{x \to a} g(x) = M$. We want to show that $M \geq 0$. To do this we show that the assumption $M < 0$ leads to a contradiction.

If $M < 0$, then $-M > 0$ and

$$\lim_{x \to a} \left(-g(x)\right) = -M > 0 .$$

Therefore,

$$-g(x) > 0 \qquad \text{(Lemma 3-4)}$$

or

$$g(x) < 0$$

contradicting the hypothesis that $g(x) \geq 0$. Thus, $M \geq 0$.

4. Prove the corollaries to Theorem 3-4e.

(a) Corollary 1. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ where $M \neq 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$.

Proof. Since $M \neq 0$, by Theorem 3-4e, we have

$$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M} .$$

From Theorem 3-4d, we have

$$\begin{aligned} \lim_{x \to a} \frac{f(x)}{g(x)} &= \lim_{x \to a} \left(f(x) \cdot \frac{1}{g(x)}\right) \\ &= \left(\lim_{x \to a} f(x)\right) \cdot \left(\lim_{x \to a} \frac{1}{g(x)}\right) \\ &= L \cdot \frac{1}{M} \\ &= \frac{L}{M} . \end{aligned}$$

(b) Corollary 2. If p and q are polynomials, and if $q(a) \neq 0$, then

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} .$$

Proof. By the corollary to Theorem 3-4d,

$$\lim_{x \to a} p(x) = p(a) \quad \text{and} \quad \lim_{x \to a} q(x) = q(a) .$$

Since $q(a) \neq 0$,

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \qquad \text{(Corollary 1 to Theorem 3-4e).}$$

5. **Find the following limits, giving at each step the theorem on limits which justifies it.**

(a) $\lim_{x\to 3} (2 + x) = \lim_{x\to 3} 2 + \lim_{x\to 3} x$ Theorem 3-4c

$= 2 + 3$ Theorem 3-4a
Example 3-3a

$= 5 .$

(b) $\lim_{x\to -1} (5x - 2) = \lim_{x\to -1} 5x + \lim_{x\to -1} (-2)$ Theorem 3-4c

$= -5 + (-2)$ Theorem 3-4b,
Example 3-3a,
Theorem 3-4a

$= -7 .$

(c) $\lim_{x\to 0} (\frac{a}{1 + |x|} - b\sqrt{|x|})$, where a and b are constants.

$\lim_{x\to 0} (\frac{a}{1 + |x|} - b\sqrt{|x|}) = \lim_{x\to 0} \frac{a}{1 + |x|} + \lim_{x\to 0} (-b\sqrt{|x|})$ Theorem 3-4c

$= a \lim_{x\to 0} \frac{1}{1 + |x|} - b \lim_{x\to 0} \sqrt{|x|}$ Theorem 3-4b

$= a \cdot 1 - b \cdot 0$ Example 3-3b,
Theorem 3-4e
Example 3-3c

$= a .$

(d) $\lim_{x\to a} (x^3 + ax^2 + a^2x + a^3)$, where a is constant.

$\lim_{x\to a} (x^3 + ax^2 + a^2x + a^3) = a^3 + a \cdot a^2 + a^2 \cdot a + a^3$ Corollary to
Theorem 3-4d

$= 4a^3 .$

6. Find the following limits, giving at each step the theorem which justifies it.

(a) $\lim_{x\to 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x\to 1} \frac{(x - 1)(x^2 + x + 1)}{(x - 1)(x + 1)}$

$= \lim_{x\to 1} (\frac{x - 1}{x - 1}) \cdot \lim_{x\to 1} (\frac{x^2 + x + 1}{x + 1})$ Theorem 3-4d

$= 1 \cdot \frac{3}{2}$ Corollary 2 to
Theorem 3-4e,
Corollary to
Theorem 3-4d

$= \frac{3}{2} .$

(b) $$\lim_{x \to 3} \frac{x^2 - 9}{x^3 - 27} = \lim_{x \to 3} \frac{(x - 3)(x + 3)}{(x - 3)(x^2 + 3x + 9)}$$

$$= \lim_{x \to 3} \left(\frac{x - 3}{x - 3}\right) \cdot \lim_{x \to 3}\left(\frac{x + 3}{x^2 + 3x + 9}\right)$$ Theorem 3-4d

$$= 1 \cdot \frac{6}{27}$$ Corollary 2 to Theorem 3-4e, Corollary to Theorem 3-4d

$$= \frac{2}{9} .$$

7. Find $\lim_{x \to 1} \frac{x^n - 1}{x - 1}$, for n a positive integer. Verify first that

$$\frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \dots + x + 1 \quad (x \neq 1) .$$

The verification of the division is done by simple algebraic methods.

Now let $p(x) = x^{n-1} + x^{n-2} + \dots + x + 1$.

$$\lim_{x \to 1} p(x) = p(1)$$ Corollary to Theorem 3-4d

$$= n .$$

8. Determine whether the following limits exist and, if they do exist, find their values.

(a) $\lim_{x \to 1} \frac{1 + \sqrt{x}}{1 - x}$ does not exist.

$\lim_{x \to 1} \frac{1 + \sqrt{x}}{1 - x} = \lim_{x \to 1} \frac{1}{1 - \sqrt{x}}$ which does not exist.

(b) $\lim_{x \to a} (x^n - a^n)$, n a positive integer, a constant.

Let $p(x) = x^n - a^n$.

$$\lim_{x \to a} p(x) = p(a)$$

$$= a - a$$

$$= 0 .$$

(c) $\lim\limits_{x \to -1} \dfrac{\sqrt{2 + x} + 1}{x + 1}$

$\lim\limits_{x \to -1} (\sqrt{2 + x} + 1) = 2$ and $\lim\limits_{x \to -1} (x + 1) = 0$, hence $\lim\limits_{x \to -1} \dfrac{\sqrt{2 + x} + 1}{x + 1}$ does not exist.

(d)
$$\lim_{x \to 1} \frac{(x - 2)(\sqrt{x} - 1)}{x^2 + x - 2} = \lim_{x \to 1} \frac{(x - 2)(\sqrt{x} - 1)}{(x + 2)(x - 1)}$$
$$= \lim_{x \to 1} \frac{(x - 2)(\sqrt{x} - 1)}{(x + 2)(\sqrt{x} + 1)(\sqrt{x} - 1)}$$
$$= \lim_{x \to 1} \frac{x - 2}{(x + 2)(\sqrt{x} + 1)} = -\frac{1}{6} .$$

(e) $\lim\limits_{x \to 1} \dfrac{1 - \sqrt{x}}{1 - x} = \lim\limits_{x \to 1} \dfrac{1}{1 + \sqrt{x}} = \dfrac{1}{2}$.

9. Using the algebra of limits show that $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} = L$ if and only if $\lim\limits_{x \to a} \dfrac{f(x) - f(a) - L(x - a)}{|x - a|} = 0$.

First, we note that $\lim\limits_{x \to a} g(x) = 0$ if and only if $\lim\limits_{x \to a} |g(x)| = 0$, and also that $|g(x)| = \big| |g(x)| \big|$.

Let
$$A = \frac{f(x) - f(a)}{x - a} - L ,$$
and
$$B = \frac{f(x) - f(a) - L(x - a)}{|x - a|} .$$
Then,
$$|A| = |B| .$$

Part 1. Assume $\lim\limits_{x \to a} B = 0$. This implies that $\lim\limits_{x \to a} |A| = 0$ and, thus,
$$\lim_{x \to a} A = 0 \quad \text{or} \quad \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = L .$$

Part 2. Assume $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} = L$. This implies that $\lim\limits_{x \to a} A = 0$ and, thus
$$\lim_{x \to a} |B| = 0 \quad \text{and} \quad \lim_{x \to a} B = 0 .$$

10. Assume $\lim\limits_{x \sim 0} \sin x = 0$ and $\lim\limits_{x \sim 0} \cos x = 1$. Find each of the following limits, if the limit exists, giving at each step the theorem on limits which justifies it.

(a)
$$\begin{aligned}\lim_{x \sim 0} \sin^3 x &= \left(\lim_{x \sim 0} \sin^2 x\right) \cdot \left(\lim_{x \sim 0} \sin x\right) && \text{Theorem 3-4d}\\ &= \left(\lim_{x \sim 0} \sin x \cdot \lim_{x \sim 0} \sin x\right) \cdot \left(\lim_{x \sim 0} \sin x\right) && \text{Theorem 3-4d}\\ &= 0 .\end{aligned}$$

(b)
$$\begin{aligned}\lim_{x \sim 0} \tan x &= \lim_{x \sim 0} \frac{\sin x}{\cos x}\\ &= \frac{\lim\limits_{x \sim 0} \sin x}{\lim\limits_{x \sim 0} \cos x} = 0 . && \text{Corollary 1 to Theorem 3-4e}\end{aligned}$$

(c)
$$\begin{aligned}\lim_{x \sim 0} \sin 2x &= \lim_{x \sim 0} (2 \sin x \cos x)\\ &= 2 \cdot \left(\lim_{x \sim 0} \sin x \cos x\right) && \text{Theorem 3-4b}\\ &= 2 \cdot \left(\lim_{x \sim 0} \sin x\right) \cdot \left(\lim_{x \sim 0} \cos x\right) && \text{Theorem 3-4d}\\ &= 0 .\end{aligned}$$

(d)
$$\begin{aligned}\lim_{x \sim 0} \frac{\sin x}{\tan x} &= \lim_{x \sim 0} \frac{\sin x \cdot \cos x}{\sin x}\\ &= \lim_{x \sim 0} \frac{\sin x}{\sin x} \cdot \lim_{x \sim 0} \cos x && \text{Theorem 3-4d}\\ &= 1 .\end{aligned}$$

(e)
$$\begin{aligned}\lim_{x \sim 0} \frac{1 - \cos x}{\sin x} &= \lim_{x \sim 0} \frac{1 - \cos x}{\sin x} \cdot \frac{1 + \cos x}{1 + \cos x} && \text{Theorem 3-4d}\\ &= \lim_{x \sim 0} \frac{\sin^2 x}{\sin x} \cdot \lim_{x \sim 0} \frac{1}{1 + \cos x} && \text{Theorem 3-4d}\\ &= 0 \cdot \frac{1}{2} = 0 .\end{aligned}$$

(f) $$\lim_{x\sim 0}\frac{\cos 2x}{\cos x + \sin x} = \lim_{x\sim 0}\frac{\cos^2 x - \sin^2 x}{\cos x + \sin x}$$

$$= \lim_{x\sim 0}(\cos x - \sin x) \cdot \lim_{x\sim 0}\frac{\cos x + \sin x}{\cos x + \sin x}$$

Theorem 3-4d

$$= \left[\lim_{x\sim 0}\cos x - \lim_{x\sim 0}\sin x\right] \cdot 1$$ Theorem 3-4c

$$= 1 .$$

11. Prove the corollaries to Theorem 3-4f.

(a) Corollary 1 (Sandwich Theorem). If $h(x) \leq f(x) \leq g(x)$ in some deleted neighborhood of a , and if $\lim_{x\sim a} h(x) = K$ and $\lim_{x\sim a} g(x) = M$, then, if $\lim_{x\sim a} f(x)$ exists, $K \leq \lim_{x\sim a} f(x) \leq M$.

Pooof. Since $f(x) \leq g(x)$ in a deleted neighborhood of a , we have, by Theorem 3-4f, that

$$\lim_{x\sim a} f(x) \leq \lim_{x\sim a} g(x) .$$

Again, since $h(x) \leq f(x)$ in a deleted neighborhood of a , we have, by Theorem 3-4f, that

$$\lim_{x\sim a} h(x) \leq \lim_{x\sim a} f(x) .$$

Thus,

$$\lim_{x\sim a} h(x) \leq \lim_{x\sim a} f(x) \leq \lim_{x\sim a} g(x)$$

or

$$K \leq \lim_{x\sim a} f(x) \leq M .$$

(b) Corollary 2 (Squeeze Theorem). (Hint: Prove $\lim_{x\sim a} f(x)$ exists.)

Because the proof of the Squeeze Theorem does not follow immediately from Theorem 3-4f it is given in Section TC3-4 immediately preceding Solutions Exercises 3-4.

A12. For what integral values of m and n does $\lim_{x\to -a} \frac{x^m + a^m}{x^n + a^n}$ exist? Find the limit for these cases.

Part 1. For $a = 0$:

(i) if $m > n$,

$$\lim_{x\to 0} \frac{x^m}{x^n} = \lim_{x\to 0} x^{m-n} = 0 \quad ;$$

(ii) if $m = n$,

$$\lim_{x\to 0} \frac{x^m}{x^n} = \lim_{x\to 0} 1 = 1 \quad ;$$

(iii) if $m < n$, $\lim_{x\to 0} \frac{x^m}{x^n}$ does not exist.

Part 2. For $a \neq 0$:

$$\lim_{x\to -a} (x^p + a^p) = \begin{cases} 2a^p \ , & p \ \text{even} \\ 0 \ , & p \ \text{odd} \ . \end{cases}$$

(i) Thus, for $a \neq 0$ and n even,

$$\lim_{x\to -a} \frac{x^m + a^m}{x^n + a^n} = \frac{\lim_{x\to -a} (x^m + a^m)}{\lim_{x\to -a} (x^n + a^n)}$$

$$= \begin{cases} \frac{2a^m}{2a^n} = a^{m-n} \ , & m \ \text{even} \\ \frac{0}{2a^n} = 0 \ , & m \ \text{odd} \ . \end{cases}$$

(ii) For $a \neq 0$, n odd, m even, $\lim_{x\to -a} \frac{x^m + a^m}{x^n + a^n}$ does not exist.

(iii) For $a \neq 0$, n odd, m odd,

$$\lim_{x \to -a} \frac{x^m + a^m}{x^n + a^n} = \lim_{x \to -a} \frac{(x + a)(x^{m-1} - ax^{m-2} + \dots + a^{m-1})}{(x + a)(x^{n-1} - ax^{n-2} + \dots + a^{n-1})}$$

$$= \frac{ma^{m-1}}{na^{n-1}}$$

$$= \frac{m}{n} a^{m-n} .$$

13. Prove that if $\lim_{x \to a} f(x) = 0$ and $g(x)$ is bounded in a neighborhood of $x = a$, then $\lim_{x \to a} f(x) \cdot g(x) = 0$.

Proof. Since $g(x)$ is bounded there exists a positive number M such that

$$-M < g(x) < M$$

in a neighborhood of $x = a$. Consequently,

$$-M \cdot |f(x)| \leq f(x)g(x) \leq M \cdot |f(x)|$$

in a neighborhood of $x = a$.

By the Squeeze Theorem,

$$\lim_{x \to a} f(x) \cdot g(x) = 0 .$$

14. (a) Verify that if $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists and if $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} f(x) = 0$.

$$\lim_{x \to a} \frac{f(x)}{g(x)} \cdot \lim_{x \to a} g(x) = \lim_{x \to a} \left(\frac{f(x)}{g(x)} \cdot g(x) \right)$$

or

$$0 = \lim_{x \to a} f(x) .$$ by Theorem 3-4d.

(b) Give examples of functions f and g for which $\lim_{x \sim a} f(x) = 0$ and $\lim_{x \sim a} f(x) = 0$ yet the limit of their quotient does not exist.

If $f(x) = x^m$ and $g(x` = x^n$, $m < n$, $\frac{f(x)}{g(x)}$ does not have a limit at 0 . Of course, many other examples exist.

Λ15. Prove that if $\lim_{x \sim a} g(x) = 0$ and $\lim_{x \sim a} f(x)$ does not exist, then the limit of the quotient $\frac{f(x)}{g(x)}$ does not exist.

If $\lim_{x \sim a} \frac{f(x)}{g(x)}$ does exist and $\lim_{x \sim a} g(x) = 0$, then $\lim_{x \sim a} f(x) = 0$ by by Number 14a. Contradiction.

16. The <u>right-hand</u> limit at a point $P(p, f(p))$ of a function is the limit of the function at the point P for a right-hand domain $(p, p + \delta)$. Similarly, for the left-hand limit, the domain is restricted to $(p - \delta, p)$. We denote them symbolically by $\lim_{x \sim p^+} f(x)$ and $\lim_{x \sim p^-} f(x)$, respectively.

In particular, $\lim_{x \sim 2^+} [x] = 2$, $\lim_{x \sim 2^-} [x] = 1$. Determine the indicated limits, if they exist, of the following:

(a) $\lim_{x \sim 2^+} \frac{[x]^2 - 4}{x^2 - 4}$.

For $x \in (2, 2 + \delta)$, $0 < \delta < 1$,

$$[x]^2 - 4 = ([x] - 2)([x] + 2)$$
$$= 0 \cdot 4 = 0 \; .$$

Thus

$$\lim_{x \sim 2^+} \frac{[x]^2 - 4}{x^2 - 4} = \lim_{x \sim 2^+} 0 = 0 \; .$$

(b) $\lim_{x \sim 2^-} \frac{[x]^2 - 4}{x^2 - 4}$.

For $x \in (2 - \delta, 2)$, $0 < \delta < 1$,.

$$[x]^2 - 4 = (-1)(3) = -3 \; .$$

Thus

$$\lim_{x \sim 2^-} \frac{[x]^2 - 4}{x^2 - 4} = \lim_{x \sim 2^-} \frac{3}{x^2 - 4}$$

does not exist.

(c) $\lim_{x \to 3^+} (x - 2 + [2 - x] - [x])$.

For $x \in (3, 3 + \delta)$, $0 < \delta < 1$,

$$[2 - x] = -2 \text{ and } [x] = 3 .$$

Thus

$$\begin{aligned}\lim_{x \to 3^+}(x - 2 + [2 - x] - [x]) &= \lim_{x \to 3^+}(x - 2 - 2 - 3) \\ &= -4 .\end{aligned}$$

(d) $\lim_{x \to 3^-}(x - 2 + [2 - x] - [x])$.

For $x \in (3 - \delta, 3)$, $0 < \delta < 1$,

$$[2 - x] = -1 \text{ and } [x] = 2 .$$

Thus,

$$\begin{aligned}\lim_{x \to 3^-}(x - 2 + [2 - x] - [x]) &= \lim_{x \to 3^-}(x - 2 - 1 - 2) \\ &= -2 .\end{aligned}$$

(e) $\lim_{x \to 0^+}\left(\frac{x}{a}\left[\frac{b}{x}\right] - \frac{b}{x}\left[\frac{x}{a}\right]\right)$, $a > 0$, $b > 0$.

Since $b > 0$, we can write $\frac{b}{x} = n + r$, where $n \leq \frac{b}{x}$ is a non-negative integer and $0 \leq r < 1$. Thus $\left[\frac{b}{x}\right] = n$, $x = \frac{b}{n + r}$, whence

$$\begin{aligned}\frac{x}{a}\left[\frac{b}{x}\right] &= \frac{bn}{a(n + r)} \\ &= \frac{b}{a(1 + \frac{r}{n})} .\end{aligned}$$

As $x \to 0^+$, n increases without bound and $\frac{r}{n} \to 0$. Since $a > 0$, $\left[\frac{x}{a}\right] = 0$ for $0 < x < a$. Thus,

$$\begin{aligned}\lim_{x \to 0^+}\left(\frac{x}{a}\left[\frac{b}{x}\right] - \frac{b}{x}\left[\frac{x}{a}\right]\right) &= \lim_{x \to 0^+}\frac{x}{a}\left[\frac{b}{x}\right] - \lim_{x \to 0^+}\frac{b}{x}\left[\frac{x}{a}\right] \\ &= \frac{b}{a} - \lim_{x \to 0^+} 0 \\ &= \frac{b}{a} .\end{aligned}$$

(f) $\lim_{x \to 0^-}\left(\frac{x}{a}\left[\frac{b}{x}\right] - \frac{b}{x}\left[\frac{x}{a}\right]\right)$, $a > 0$, $b > 0$.

The first term is similar to the first term in (e) except that n is a negative integer. Since $a > 0$, $\left[\frac{x}{a}\right] = -1$ for $0 < |x| < a$; however, $\left|\frac{b}{x}\right|$ increases without bound as $x \to 0^-$. Thus $\lim_{x \to 0^-}\left(\frac{x}{a}\left[\frac{b}{x}\right] - \frac{b}{x}\left[\frac{x}{a}\right]\right)$ does not exist.

(g) $\lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{4 + \sqrt{x}} - 2}$.

$$\lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{4 + \sqrt{x}} - 2} = \lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{4 + \sqrt{x}} - 2} \cdot \frac{\sqrt{4 + \sqrt{x}} + 2}{\sqrt{4 + \sqrt{x}} + 2}$$

$$= \lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{x}} \cdot (\sqrt{4 + \sqrt{x}} + 2)$$

$$= 4 .$$

TC3-5. The Idea of Continuity.

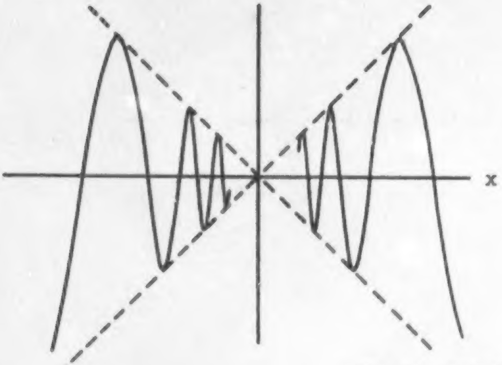
The idea of continuity seems to be intuitively clear. A function that can be represented graphically as an unbroken curve is, of course, continuous. On the other hand, there are continuous functions with such bizarre behavior that their graphs cannot be accurately described by a drawing.

Example TC3-5. Consider the function f given by

$$f(x) = \begin{cases} x \sin \frac{1}{x} , & x \neq 0 \\ 0 , & x = 0 \end{cases} .$$

(See Figure TC3-5a.)

Since $\lim_{x \to 0} f(x) = f(0) = 0$, f is continuous at 0; however, we cannot draw the graph of f in the vicinity of the origin because f oscillates infinitely often in any neighborhood of $x = 0$.



$$f : x \longrightarrow \begin{cases} x \sin \frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

Figure TC3-5a

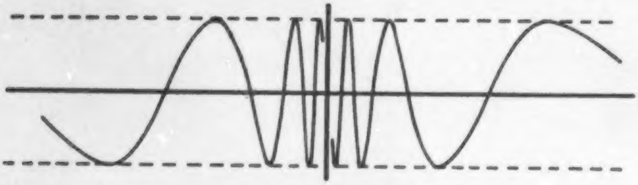
In this section, we are only concerned with local properties of a function. The concept of continuity at a point involves the behavior of a function in a neighborhood of the given point; hence the function must be defined for some x within every δ-neighborhood of the point. We can think of Definition 3-5 in another way: f is continuous at a if, corresponding to each ϵ-neighborhood of $f(a)$, say N_ϵ, there exists a δ-neighborhood of a whose image is contained in N_ϵ. (Note that a is not deleted from the δ-neighborhood of a.)

There are various kinds of discontinuities of a function. It is possible for a function to be discontinuous at all points. For example, the function given by

$$f(x) = \begin{cases} 0 & , x \text{ rational} \\ 1 & , x \text{ irrational} \end{cases}$$

is not continuous for any x (at x rational or at x irrational).

The function $f : x \longrightarrow \frac{1}{x}$ is discontinuous at $x = 0$. This discontinuity cannot be removed by assigning a value to $f(x)$ at $x = 0$ because as x approaches 0 the value of $f(x) = \frac{1}{x}$ increases without bound; i.e., $\lim_{x \to 0} \frac{1}{x}$ does not exist. The function $f : x \longrightarrow \sin \frac{1}{x}$ is also discontinuous at $x = 0$. (See Figure TC3-5b.) The discontinuity is not removable because near $x = 0$ the function oscillates between 1 and -1, i.e., $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.



$f : x \longrightarrow \sin \frac{1}{x}$

Figure TC3-5b

If a function g is not continuous at a but is defined for all values of x in a neighborhood of a (including a) and if $\lim_{x \to a} g(x) = L$, then it is possible to remove the discontinuity by reassigning a different functional value at a, namely $g(a) = L$.

Solutions Exercises 3-5

1. Use the formal definition of continuity to show that each of the following functions is continuous at $x = 1$. (See Exercises 3-3, Nos. 4d and 4g.)

(a) $f : x \longrightarrow \dfrac{x + 1}{x^2 + 1}$

In Exercises 3-3, Number 4d, we proved

$$\lim_{x \to 1} \frac{x + 1}{x^2 + 1} = 1$$

whence, since $f(1) = 1$, f is continuous at 1.

(b) $g : x \longrightarrow \dfrac{4x^2 - 3x - 1}{x + 2}$

In Exercises 3-3, Number 4g, we proved

$$\lim_{x \to 1} \frac{4x^2 - 3x - 1}{x + 2} = 0 ;$$

whence, since $g(1) = 0$, g is continuous at 1.

2. For what value(s) of x is each of the following functions discontinuous? Justify your answer.

   (a) $f : x \longrightarrow \frac{x}{x}$

   Since $f(x)$ is not defined here, f is discontinuous at 0.

   (b) $f : x \longrightarrow \frac{x^2}{x + 1}$

   The function f is discontinuous at -1; f is not defined at $x = -1$ and, in addition, $\lim\limits_{x \to -1} f(x)$ does not exist.

3. Which of the following functions are discontinuous at $x = -1$? Justify your answer.

   (a) $f : x \longrightarrow \frac{x + 1}{x^3 + 1}$ is discontinuous at -1 since $f(-1)$ is not defined.

   (b) $g : x \longrightarrow \frac{1}{1 + x^2}$ is continuous at -1 since $\lim\limits_{x \to -1} g(x) = g(-1) = \frac{1}{2}$.

   (c) $h : x \longrightarrow \frac{1}{1 - x^2}$ is discontinuous at -1 since $h(-1)$ is not defined (and $\lim\limits_{x \to -1} h(x)$ does not exist).

4. Discuss the points of discontinuity of $f : x \longrightarrow [x] + [-x]$. (See Exercises 3-1, No. 1.)

   For all integers n, $\lim\limits_{x \to n} f(x)$ does not exist; hence f is discontinuous at all points $x = n$.

5. Prove that $f : x \longrightarrow x - [x]$ is continuous for every x which is not an integer and discontinuous for integral values of x.

   For any x such that

   $$n < x < n + 1 \qquad (n \text{ an integer}),$$

   $f(x) = x - n$; thus f is continuous.

   Since $f(n + 1) = 0$ and

   $$\lim_{x \to n+1} (x - n) = 1,$$

   f is discontinuous at integral values of x.

6. For each of the following functions define a new function which agrees with the given one for $x \neq a$ and is continuous at $x = a$.

   (a) $f : x \longrightarrow \frac{x^3 - 1}{x^2 - 1}$, $a = 1$.

   $$g : x \longrightarrow \frac{x^2 + x + 1}{x + 1}$$

   (b) $f : x \longrightarrow \frac{x^2 - 4}{x^3 - 8}$, $a = 2$.

   $$g : x \longrightarrow \frac{x + 2}{x^2 + 2x + 4}$$

   (c) $f : x \longrightarrow \frac{x^n - 1}{x - 1}$, $a = 1$, n an integer.

   $$g : x \longrightarrow x^{n-1} + x^{n-2} + \dots + x + 1$$

   (d) $f : x \longrightarrow \frac{1 - \sqrt{x}}{1 - x}$, $a = 1$.

   $$g : x \longrightarrow \frac{1}{1 + \sqrt{x}}$$

   (e) $f : x \longrightarrow \frac{(x - 2)(\sqrt{x} - 1)}{x^2 + x - 2}$, $a = 1$.

   $$g : x \longrightarrow \frac{x - 2}{(x + 2)(1 + \sqrt{x})}$$

7. For each of the following functions, if possible, define a new function which agrees with the given one for $x \neq 0$ and is continuous at $x = 0$. If this is not possible, state why.

   (a) $f : x \longrightarrow \frac{x}{x}$.

   $$g : x \longrightarrow 1.$$

   (b) $f : x \longrightarrow \frac{x}{|x|}$.

   Impossible, since $\lim\limits_{x \sim 0} f(x)$ does not exist.

   (c) $f : x \longrightarrow \frac{1}{|x|}$.

   Impossible, since $\lim\limits_{x \sim 0} f(x)$ does not exist.

(d) $f : x \longrightarrow x - [x]$.

Impossible, since $\lim_{x \sim 0} f(x)$ does not exist.

8. For each of the following functions show that no function which agrees with the given one for $x \neq a$ can be so defined as to be continuous at $x = a$.

(a) $f : x \longrightarrow \frac{1 + \sqrt{x}}{1 - x}$, $a = 1$.

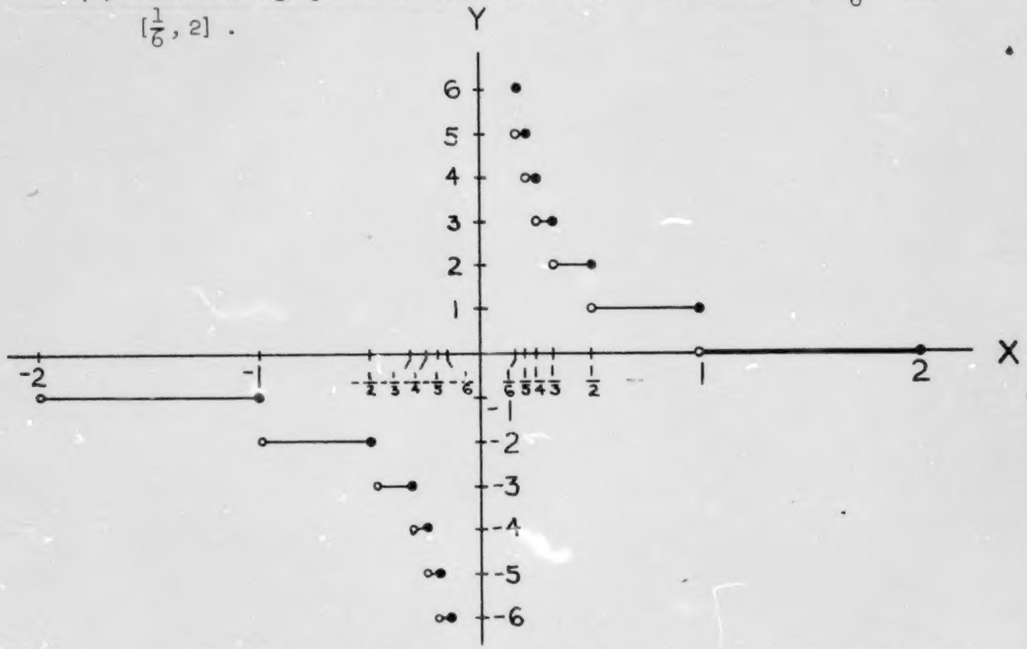
$\lim_{x \sim 1} f(x)$ does not exist (Exercises 3-4, No. 8a).

(b) $g : x \longrightarrow \frac{\sqrt{2 + x} + 1}{x + 1}$, $a = -1$.

$\lim_{x \sim -1} g(x)$ does not exist (Exercises 3-4, No. 8c).

9. For $x \neq 0$, let $f(x) = \left[\frac{1}{x}\right]$.

(a) Sketch the graph of f over the closed intervals $[-2, -\frac{1}{6}]$ and $[\frac{1}{6}, 2]$.



(b) What happens to $f(x)$ as x approximates 0 by positive values? by negative values?

$|f(x)|$ increases without bound as x approaches 0 . (For $x < 0$, $f(x) < 0$.)

(c) Can you define a function which agrees with f for $x \neq 0$ and which is continuous at $x = 0$?

No. $\lim\limits_{x \sim 0} f(x)$ does not exist.

10. If f is an increasing function whose domain is the set of all real numbers, and if f is not continuous at a , what can you say of

$$\lim_{x \sim a} |f(x) - f(a)| \ ?$$

If $\lim\limits_{x \sim a} f(x) = L$ $(L \neq a)$, then $\lim\limits_{x \sim a} |f(x) - f(a)| = |L - f(a)|$.

If $\lim\limits_{x \sim a} f(x)$ does not exist, $\lim\limits_{x \sim a} |f(x) - f(a)|$ generally does not exist. For an example where it does exist, consider the following. Let $a = 0$ and

$$f(x) = \begin{cases} 3 , & x \text{ irrational} \\ 1 , & x \text{ rational} , \quad x \neq 0 \\ 2 , & x = 0 . \end{cases}$$

Then $\lim\limits_{x \sim a} |f(x) - f(0)| = 1$.

11. For every real x , let $N(x)$ denote the number of distinct real square roots of x , i.e., the number of distinct real solutions of $y^2 = x$. Where does N have a limit? What is the limit? Where is N continuous? Let $P(x) = \left(N(x) - 1\right)^2$. Where does P have a limit? What is the limit? Where is P continuous? How does P differ from the function $f : x \longrightarrow 1$? from the function $g : x \longrightarrow \frac{x}{x}$?

First, let us describe N .

For $x < 0$, $y^2 = x$ has no real solution, so $N(x) = 0$;

at $x = 0$, $y^2 = 0$ has one real solution, so $N(x) = 1$;

for $0 < x$, $y^2 = x$ has two real solutions, so $N(x) = 2$.

N has a limit at every $x_0 \neq 0$. If $x_0 < 0$, $\lim\limits_{x \sim x_0} N(x) = 0 = N(x_0)$. If $0 < x_0$, $\lim\limits_{x \sim x_0} N(x) = 2 = N(x_0)$. Thus $N(x)$ is continuous for all $x \neq 0$. Since $\lim\limits_{x \sim 0} N(x)$ does not exist, $N(x)$ is not continuous at 0 .

$P(x) = \left(N(x) - 1\right)^2$. Then

$$P(x) = \begin{cases} 1 , & \text{if } x < 0 \\ 0 , & \text{if } x = 0 \\ 1 , & \text{if } 0 < x . \end{cases}$$

Since $\lim_{x \to x_0} P(x) = P(x_0)$, $x_0 \neq 0$, P is continuous at all $x_0 \neq 0$.

Since $\lim_{x \to 0} P(x) = 1 \neq P(0)$, P is discontinuous at 0 .

$P(0) \neq f(0)$. P is defined at $x = 0$, whereas g is not.

12. Each of the functions f , g , and h is defined for all real x . Which of the functions are not continuous at 0 ?

$$f(x) = \begin{cases} 0 , & x \text{ rational} \\ 1 , & x \text{ irrational} \end{cases}$$

$$g(x) = \begin{cases} 0 , & x \text{ rational} \\ x , & x \text{ irrational} \end{cases}$$

$$h(x) = \begin{cases} 1 , & x \text{ rational} \\ -1 , & x \text{ irrational} . \end{cases}$$

The functions f and h are not continuous at 0 . Since $\lim_{x \to 0} g(x) = g(0)$, g is continuous at 0 .

13. Give an example of a function which is not continuous at 0 but whose absolute value is continuous at 0 .

Consider the function h in Number 12.

Λ14. (a) Show that the function f of Number 12 is periodic and determine all possible periods.

Any rational number r is a period. This follows since $x + r$ is rational if x is rational, and $x + r$ is irrational if x is irrational. So,

$$f(x + r) = 0 = f(x) , \text{ for } x \text{ rational} ,$$
$$f(x + r) = 1 = f(x) , \text{ for } x \text{ irrational.}$$

(b) Show that every nonconstant periodic function which is continuous, at least at one point, has a fundamental (smallest) period.

The proof here is indirect. We assume that there is not a smallest period and obtain a contradiction. Without loss of generality, we can take the point of continuity at $x = 0$. Since the function is nonconstant, it must take on a value other than $f(0)$ somewhere, say at $x = a$. Let p be an arbitrarily small period; then

$$f(a) = f(a - np) , n = 1 , 2 , 3 , \ldots .$$

Since p is arbitrarily small, $a - np$ can be made arbitrarily close to 0 . This is impossible since the function is continuous at $x = 0$ (f must take on values close to $f(0)$ in the neighborhood of $x = 0$).

A15. If $f(x) = \begin{cases} \frac{1}{q} , & x \text{ rational, } x = \frac{p}{q} \text{ (} p , q \text{ relatively prime)} \\ 0 , & x \text{ irrational.} \end{cases}$

Show that f is continuous for all irrational x , and discontinuous for all rational x .

The point of this exercise is to show that continuity at a point is not the concept appropriate to our geometrical sense. It is not continuity at a point which provides a base for the calculus, but continuity on an interval. This example shows that a function may have both points of continuity and discontinuity in every interval of its domain.

Solution. Since every interval contains both rational and irrational points (Section A1-5) it is easily verified that f is discontinuous at rational points $x = \frac{p}{q}$. Namely, every neighborhood of $\frac{p}{q}$ contains at least one irrational point r ; hence,

$$|f(r) - f(\tfrac{p}{q})| = |0 - \tfrac{1}{q}| = \frac{1}{q} > \frac{1}{2q} .$$

Thus the condition for the continuity of f (Definition 3-5) is violated for $\epsilon = \frac{1}{2q}$.

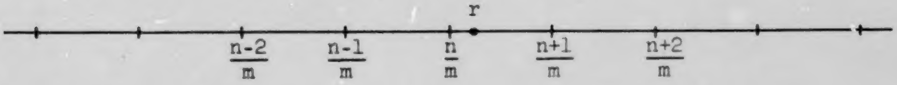
The proof of continuity at irrational points is subtler. In effect, we show that to approximate an irrational number closely by rational numbers, we cannot use rational numbers with small denominators. For the proof we observe for any real number r and fixed denominator m , there is precisely one numerator n for which

$$\frac{n}{m} \leq r < \frac{n+1}{m}$$

(namely $n = [mr]$) . For any numerator k other than n and $n + 1$ we must have

$$\left|\frac{k}{m} - r\right| > \frac{1}{m} \tag{1}$$

(see Figure). Furthermore, if $q \geq m$, it follows from (1) that $\left|\frac{k}{m} - r\right| > \frac{1}{q}$. Thus for each number $m = 1, 2, 3, \ldots, q$ there are at most two rational numbers of the form $\frac{k}{m}$ for which $\left|\frac{k}{m} - r\right| < \frac{1}{q}$, at most $2q$ such numbers in all. If r is irrational, then we let δ_q be the closest distance of approach to r by rational number with denominator no larger than q:



Figure

$$\delta_q = \min\{\left|\frac{k}{m} - r\right| ;\ k \text{ any integer, } 1 \leq m \leq q\}$$

Since $\frac{k}{m}$ can be restricted to the finite set for which $\left|\frac{k}{m} - r\right| < \frac{1}{q}$, the minimum exists. Since r is irrational, it cannot be equal to any of the members of this finite set, so $\delta_q > 0$.

To prove continuity at $x = r$, r irrational, let ϵ be any positive number and choose $q > \frac{1}{\epsilon}$. Then for any rational number $\frac{s}{t}$ satisfying

$$\left|\frac{s}{t} - r\right| < \delta_q$$

we must have $t > q$, hence

$$\begin{aligned} \left|f\left(\frac{s}{t}\right) - f(r)\right| &= \left|\frac{1}{t} - 0\right| = \frac{1}{t} \\ &< \frac{1}{q} \\ &< \epsilon . \end{aligned}$$

TC3-6. Properties of Functions Continuous at a Point.

A geometric interpretation of composition of functions serves to clarify the idea for some students. This is particularly true if the student actually obtains a sketch of the graph of the composite function fg from the graphs of f and g . The restrictions imposed upon the domain of fg become apparent.

We start with the graphs of f and g which are given. (Figure TC3-6a.)

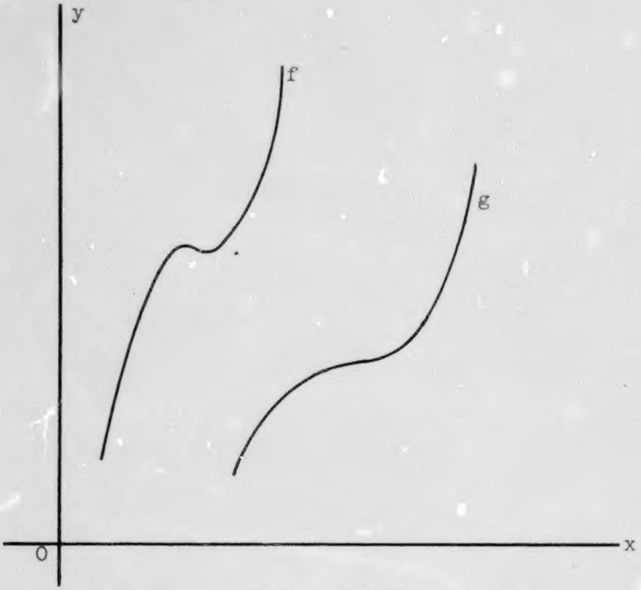


Figure TC3-6a

The line $y = x$ will be used to reflect points on the x-axis. Thus, for example, for any given point (c,d) we may locate the point $(d,0)$ which is the projection of the point (d,d) on the x-axis (Figure TC3-6b).

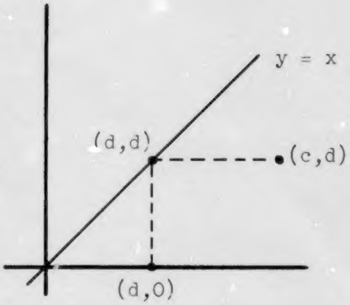


Figure TC3-6b

We select any point x_1 in the domain of g and locate the point $\left(x_1, g(x_1)\right)$ on the graph of g. Then, using the line $y = x$ as above, we locate the point $\left(g(x_1), 0\right)$ and, thereafter, the point $\left(g(x_1), f\left(g(x_1)\right)\right)$ on the graph of f (Figure TC3-6c).

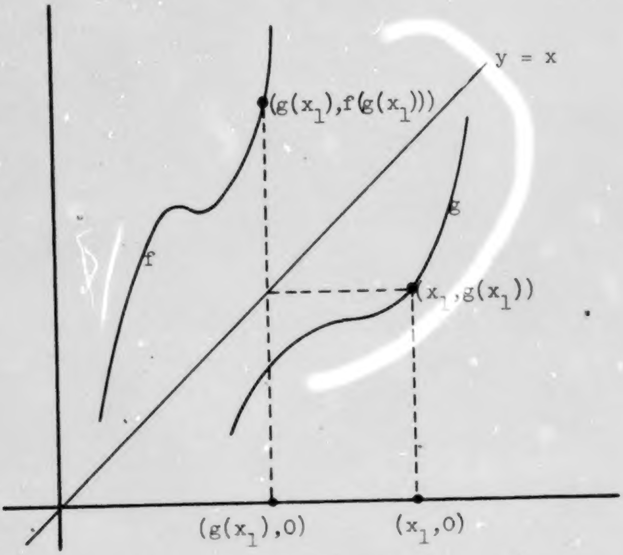


Figure TC3-6c

Finally, we locate the point of intersection of the lines $x = x_1$ and $y = f\left(g(x_1)\right)$. This point, $P\left(x_1, f\left(g(x_1)\right)\right)$, is on the graph of fg (Figure TC3-6d).

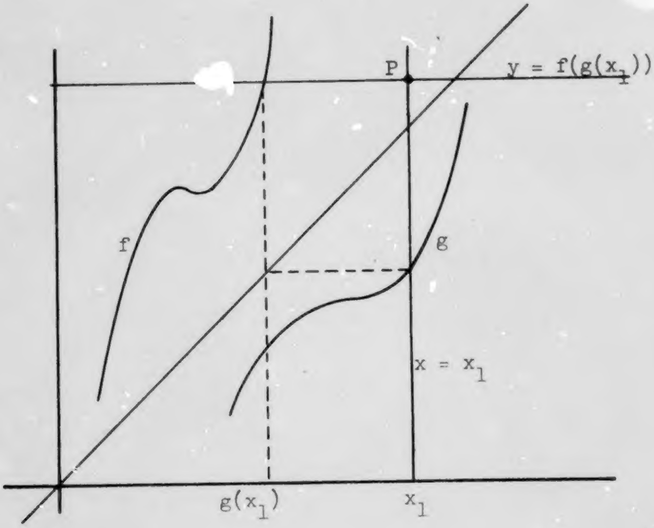


Figure TC3-6d

We observe that the points $\left(x_1, g(x_1)\right)$, $\left(g(x_1), g(x_1)\right)$, $\left(g(x_1), fg(x_1)\right)$, and $P\left(x_1, fg(x_1)\right)$ form a rectangle. The construction, then, can be described as a sliding rectangle (of changing dimensions) with one vertex on the graph of g , one vertex on the line $y = x$, one vertex on the graph of f , and the fourth vertex on the graph of fg . The construction is unique when the first three vertices are obtainable; in other words, x_1 must be in the domain of g and $g(x_1)$ must be in the domain of f . A sketch of the graph is shown in Figure TC3-6e. The domain of fg (indicated on the x-axis) is the closed interval $[a,b]$. Note that the interval $[a,b]$ is in the domain of g and the interval $[g(a), g(b)]$ is in the domain of f .

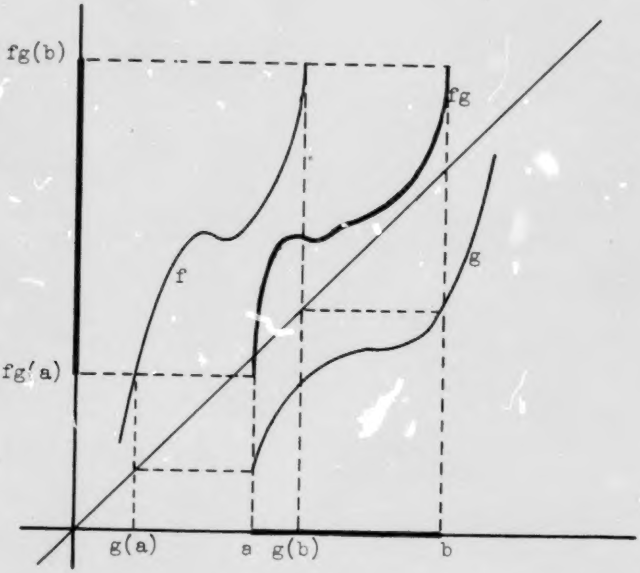
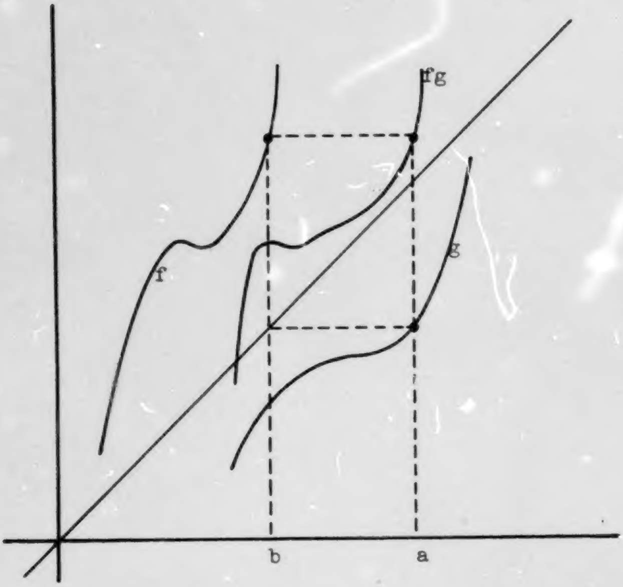
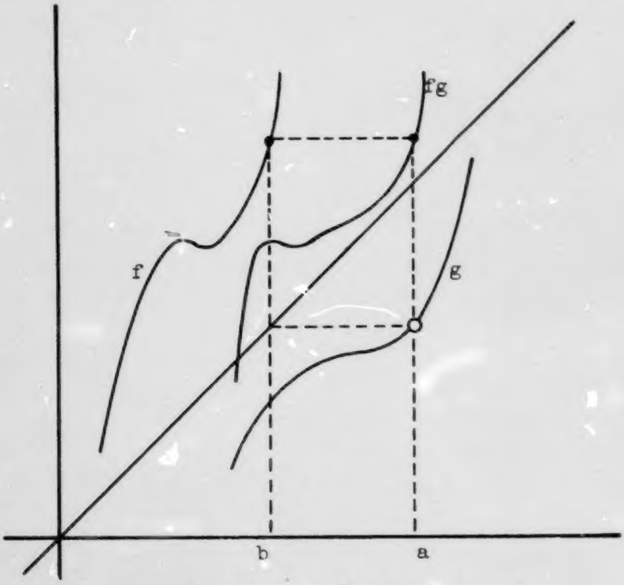


Figure TC3-6e

In Section 3-4 we proved that the existence of the limit is preserved under the functional operations of addition, multiplication, and division (under certain restrictions). In this section we prove that composition of functions preserves continuity, and hence the existence of the limit. If we compare the statements in Theorem 3-6e and Exercises 3-6b, No. 5, we note that the theorem includes the additional hypothesis that g is defined at a and $g(a) = b$. This distinction is shown in Figures 3-6f and 3-6g.



Theorem 3-6e
Figure TC3-6f



Exercises 3-6b, No. 5
Figure TC3-6g

We observe that, in either case, the function fg is continuous at $x = a$ under the hypothesis that $\lim_{x \sim a} g(x) = b$ and f is continuous at $x = b$. It is immaterial whether g is defined at $x = a$ or, if defined, what its value is at $x = a$. In other words, we may delete the additional hypothesis of Theorem 3-6e (g is defined at a and $g(a) = b$) without disturbing the continuity of fg .

The theorems of this section provide means of establishing continuity of functions constructed by rational operations and composition. Frequently, it is possible, and more convenient, to establish continuity by establishing differentiability.

At present, we are only concerned with local properties of a function.

Solutions Exercises 3-6a

1. Prove that $f : x \longrightarrow x^2$ is continuous at $x = a$, where a is any real number.

   $g : x \longrightarrow x$ is continuous at a , for every real number a (since $\lim_{x \sim a} g(x) = g(a)$ for every real a). Thus, $f(x) = g(x) \cdot g(x)$ is continuous at every real a by Theorem 3-6b.

2. (a) Prove Theorem 3-6b using the limit theorems as in the proof of Theorem 3-6a.

   Theorem 3-6b. If the functions f and g are continuous at $x = a$, then so is the function h defined by $h(x) = f(x) \cdot g(x)$. That is, the product of two functions continuous at $x = a$ is also continuous there.

$$\lim_{x \sim a} h(x) = \lim_{x \sim a} [f(x)g(x)] \qquad \text{Definition of } h(x)$$

$$= [\lim_{x \sim a} f(x)][\lim_{x \sim a} g(x)] \qquad \text{Theorem 3-4(d)}$$

$$= [f(a)][g(a)] \qquad \text{Definition of continuity}$$

$$= h(a) \qquad \text{Definition of } h(x) .$$

   Hence, h is continuous at $x = a$, by the definition of continuity.

(b) Prove Theorem 3-6c in the same way.

Theorem 3-6c. If the functions f and g are continuous at $x = a$, and if $g(a) \neq 0$, then the function h defined for $g(x) \neq 0$ by $h(x) = \frac{f(x)}{g(x)}$, is continuous at $x = a$. In other words, the quotient of continuous functions is continuous if no division by zero is involved.

$\lim\limits_{x \to a} h(x) = \lim\limits_{x \to a} \frac{f(x)}{g(x)}$	Definition of h
$= \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$	Corollary 1 to Theorem 3-4e
$= \frac{f(a)}{g(a)}$	Definition 3-5
$= h(a)$	

Hence by Definition 3-5, h is continuous at a.

3. Prove Theorems 3-6b and 3-6c directly from Definition 3-5.

Theorem 3-6b. The proof is the same as that of Theorem 3-4d with L and M replaced by $f(a)$ and $g(a)$, respectively, and in the last step $f(x) \cdot g(x)$ and $f(a) \cdot g(a)$ replaced by $h(x)$ and $h(a)$, respectively.

Theorem 3-6c. The proof is the same as that of Corollary 1 to Theorem 3-4e with L and M replaced by $f(a)$ and $g(a)$, respectively, and in the last step, $\frac{f(x)}{g(x)}$ and $\frac{f(a)}{g(a)}$ replaced by $h(x)$ and $h(a)$, respectively. (See Exercises 3-4, No. 4.)

∧ 4. (a) If the function f is continuous at $x = a$ and the function g is not continuous at $x = a$, show that $f + g$ is not continuous at $x = a$.

Suppose the function $f + g$ were continuous at $x = a$. Since f is continuous at $x = a$, so is $-f$ (Theorem 3-6b). Hence, $(f + g) + (-f)$ is continuous at $x = a$, by Theorem 3-6a. But $(f + g) + (-f) = g$, so g is continuous at $x = a$; a contradiction.

(b) Can $f + g$ be continuous at $x = a$ if neither f nor g is continuous at $x = a$? Illustrate your answer by giving an example.

The given conditions do not determine whether $f + g$ is continuous at a (in general, it would not be continuous). If $f : x \longrightarrow \operatorname{sgn} x$ and $g : x \longrightarrow -\operatorname{sgn} x$, then $f + g : x \longrightarrow 0$ is continuous at a . Number 5 is another example.

(c) Repeat the above using $f \cdot g$ for $f + g$.

We first consider the continuity of $f \cdot g$ at a , given that f is continuous at a and g is not continuous at a .

Part 1. Consider the case, $f(a) \neq 0$. Suppose $f \cdot g$ is continuous at a , then the function $\frac{f \cdot g}{f}$, defined for $f(x) \neq 0$, is continuous at a (Theorem 3-6c). But $\frac{f \cdot g}{f} = g$, so g is continuous at a ; a contradiction.

Part 2. $f(a) = 0$. The given conditions do not determine whether $f \cdot g$ is continuous at a . For example, if $f : x \longrightarrow x$ and $g : x \longrightarrow \frac{1}{x^2}$, then $f \cdot g : x \longrightarrow \frac{1}{x}$ is not continuous at 0 . On the other hand, if $f : x \longrightarrow x^2$ and $g : x \longrightarrow \frac{1}{x}$, then $f \cdot g$ is continuous at 0).

Next, we consider the continuity of $f \cdot g$ at a , given that neither f nor g is continuous at a . The given conditions do not determine whether $f \cdot g$ is continuous at a . For example,

$$\text{if } f : x \longrightarrow \begin{cases} k , & x \neq 0 \\ \frac{1}{k} , & x = 0 \end{cases} \quad \text{and} \quad g : x \longrightarrow \begin{cases} \frac{1}{k} , & x \neq 0 \\ k , & x = 0 , \end{cases}$$

$k > 1$, then $f \cdot g : x \longrightarrow 1$ is continuous at 0 .

5. Determine where the function $f : x \longrightarrow [x] + \sqrt{x - [x]}$ is continuous.

f is everywhere continuous.
The functions

$g : x \longrightarrow [x]$ Example 3-5b and

$$h : x \longrightarrow \sqrt{x - [x]}$$

Example A2-1f, Corollary to Theorem 3-6d, Theorem 3-6e

are continuous for all non-integral values of x. Thus $f = g + h$ is continuous there, (Theorem 3-6a).

To verify the continuity of f for integral values of x, we show that

$$\lim_{x \to n^+} f(x) = \lim_{x \to n^-} f(x) .$$

$$\lim_{x \to n^+} f(x) = n + \sqrt{n - n} = n$$

$$\lim_{x \to n^-} f(x) = n - 1 + \sqrt{n - (n - 1)} = n .$$

Since $f(n) = n$, the result follows.

Solutions Exercises 3-6b

1. Prove that $f : x \longrightarrow |x|$ does not have a derivative at $x = 0$.

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x|}{x} = \lim_{x \to 0} \frac{x}{|x|}$$ does not exist (Exercises 3-1, No. 2c).

Λ2. Let $f : x \longrightarrow x^n$, where n is a positive integer.

(a) Use the binomial theorem to expand $(x + h)^n$.

By the binomial theorem,

$$(x + h)^n = x^n + \binom{n}{1} x^{n-1}h + \binom{n}{2} x^{n-2}h^2 + \dots + h^n .$$

(b) From the result of (a) derive a formula for

$$r(x) = \frac{f(x + h) - f(x)}{h} .$$

$$\frac{f(x + h) - f(x)}{h} = \binom{n}{1} x^{n-1} + \binom{n}{2} x^{n-2} h + \dots + h^{n-1} .$$

(c) From the result of (b) deduce that

$$m = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = nx^{n-1} .$$

State which limit theorems you are using.

Since $r(x)$ is a polynomial in h , $p(h)$, we have by Corollary to Theorem 3-4d that

$$\lim_{h \to 0} p(h) = p(0)$$
$$= nx^{n-1} .$$

(d) Use Theorem 3-6d to show that f is continuous at $x = a$, where a is any real number.

Since $f(x)$ has a derivative at $x = a$, (namely, na^{n-1}) by Theorem 3-6d f is continuous at a .

3. Let $f(x) = \sqrt{x}$, $g(x) = \frac{1}{x - 1}$, and $h(x) = \sin x$. Describe the domain of the function given by

(a) $fg(x) = \sqrt{\frac{1}{x - 1}}$; $x > 1$

(b) $gf(x) = \frac{1}{\sqrt{x} - 1}$; $x \geq 0$ and $x \neq 1$

(c) $hg(x) = \sin \frac{1}{x - 1}$; $x \neq 1$

(d) $gh(x) = \frac{1}{\sin x - 1}$; $x \neq \frac{\pi}{2}(1 + 4n)$, n an integer.

(e) $hfg(x) = \sin \sqrt{\frac{1}{x - 1}}$; $x > 1$

4. Assume that the functions $x \to \sin x$ and $x \to \cos x$ are continuous for all x . Find the discontinuities of the function given by

(a) $f(x) = \sin \frac{1}{x}$; discontinuity at 0 .

(b) $f(x) = \tan x$; discontinuities at $\frac{\pi}{2}(1 + 2n)$, n an integer.

(c) $f(x) = \dfrac{1}{4 - 3\sin^2 x}$; no discontinuities $(4 - 3\sin^2 x > 0$ for all $x)$.

(d) $f(x) = \sin\cos x$; no discontinuities.

(e) $f(x) = \tan\dfrac{x^2 - 1}{x + 1}$; discontinuities at -1 , $\frac{\pi}{2}(1 + 2n) + 1$, n an integer.

∧ (f) $f(x) = \tan\cos x - \cos\tan x$; discontinuities at $\frac{\pi}{2}(1 + 2n)$.

5. Prove, if $\lim\limits_{x\to a} g(x) = b$ and f is continuous at b , then the composite function $x \longrightarrow fg(x)$, has a limit as x approaches a and $\lim\limits_{x\to a} f\left(g(x)\right) = f\left(\lim\limits_{x\to a} g(x)\right) = f(b)$.

Set $g(x) = u$.
Then, since $\lim\limits_{x\to a} g(x) = b$, for any $\eta_1 > 0$ we have

$$|u - b| < \eta_1$$

for a sufficiently small deleted δ_1 neighborhood of a .
Also, by the continuity of f at b , for any $\epsilon > 0$,

$$|f(u) - f(b)| < \epsilon$$

for a sufficiently small η neighborhood of b . Thus, for any $\epsilon > 0$, there is a deleted δ-neighborhood of a sufficiently small to control the tolerance $\eta_1 \leq \eta$ and for this δ if $0 < |x - a| < \delta$, we have

$$\left|f\left(g(x)\right) - f(b)\right| < \epsilon .$$

Since

$$\left|f\left(g(x)\right) - f(b)\right| = \left|fg(x) - f\left(\lim_{x\to a} g(x)\right)\right| .$$

the theorem is proved.

A6. Prove that if $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$, then

$$\lim_{x\to a} \sqrt{\big(f(x)\big)^2 + \big(g(x)\big)^2} = \sqrt{L^2 + M^2} .$$

By the corollary to Theorem 3-6d, $f : x \longrightarrow \sqrt{x}$ is continuous at $x = a$ for $a > 0$. Since $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ both exist, by Theorems 3-4d and 3-4c, so does

$$\lim_{x\to a} \Big(\big(f(x)\big)^2 + \big(g(x)\big)^2\Big) .$$

By Number 5,

$$\lim_{x\to a} \sqrt{\big(f(x)\big)^2 + \big(g(x)\big)^2} = \sqrt{\lim_{x\to a} \Big(\big(f(x)\big)^2 + \big(g(x)\big)^2\Big)}$$
$$= \sqrt{L^2 + M^2} ,$$

provided $L^2 + M^2 \neq 0$.

For the case $L = M = 0$,

$$\lim_{x\to a} f(x) = \lim_{x\to a} |f(x)| = 0$$

and

$$\lim_{x\to a} g(x) = \lim_{x\to a} |g(x)| = 0 .$$

Since

$$0 \leq \sqrt{\big(f(x)\big)^2 + \big(g(x)\big)^2} \leq |f(x)| + |g(x)|$$

it follows by the Sandwich Theorem that

$$\lim_{x\to a} \sqrt{\big(f(x)\big)^2 + \big(g(x)\big)^2} = 0 .$$

TC3-7. Properties of Functions Continuous on an Interval.

Intuitively, we visualize the graph of a function f, continuous over a closed interval $[a,b]$, as a curve: we assume that if ξ is any number between $f(a)$ and $f(b)$ then the line $y = \xi$ intersects the graph of f at least once in the interval $[a,b]$. Geometrically, the assumption (the Intermediate Value Theorem) is plausible; however, the proof (which must be given in analytical terms without recourse to visualization) is not at first obvious (see Appendix 4).

The intermediate value property is not restricted to continuous functions. For example, the function given by

$$f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(see Figure TC3-5b) is not continuous at 0; however, there exists a number u in every closed interval $[a,b]$ such that $f(u)$ is between $f(a)$ and $f(b)$.

If f is continuous over $[a,b]$ with $f(a)$ negative and $f(b)$ positive (or the opposite) then the initial point of the graph is below (above) the x-axis and the endpoint is above (below) the x-axis. In intermediate (advanced) algebra we assume that the curve will cross the x-axis somewhere (at least once) in the open interval (a,b) and use this assumption as a means (the familiar "change-of-sign" test) of isolating zeros of a function. The assumption (stated below as a corollary) is an immediate consequence of the Intermediate Value Theorem.

Corollary to Theorem 3-7a. If f is continuous over the closed interval $[a,b]$ with $f(a) < 0$ and $f(b) > 0$, then there is at least one value c, $a < c < b$, for which $f(c) = 0$.

The Extreme Value Theorem specifies: when f is continuous over a closed interval, that a maximum and a minimum occur at least once, that the maximum (minimum) may occur at either endpoint, a or b, or in the open interval (a,b). We observe, then, that the property of Theorem 3-7b depends on the interval as well as the function. It involves the set $\{x : a \leq x \leq b\}$, as a whole. In this sense, it is sometimes called a global property. (It is obvious that Theorem 3-7b refers to so-called "absolute" maxima (minima) as opposed to "relative" maxima (minima). We shall not use this terminolgy in the text.)

In Section TC3-5 we considered the function f given by

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and noted that it is impossible to draw the graph in any interval containing the origin. Now we show that f is not differentiable at $x = 0$.

$$r(x) = \frac{f(x) - f(0)}{x - 0}$$

$$= \frac{x \sin \frac{1}{x}}{x} = \sin \frac{1}{x}, \quad x \neq 0 .$$

For $\frac{1}{x} = \frac{\pi}{2} + 2n\pi$; i.e., for $x = \frac{2}{\pi(1 + 4n)}$, $r(x) = 1$, (n an integer).

For $\frac{1}{x} = \frac{3\pi}{2} + 2n\pi$; i.e., for $x = \frac{2}{\pi(3 + 4n)}$, $r(x) = -1$.

From this we observe that $r(x)$ oscillates between -1 and 1 as x approaches zero. Therefore the function f is not differentiable at $x = 0$ since $\lim_{x \to 0} r(x)$ does not exist.

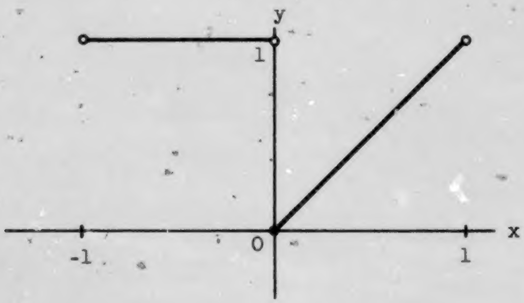
Solutions Exercises 3-7

1. Exhibit a discontinuous function for which the range of the function is an interval.

(a) Let f be defined for $-1 < x < 1$,

$$f(x) = \begin{cases} 1 & \text{for } -1 < x < 0 \\ x & \text{for } 0 \leq x < 1 . \end{cases}$$

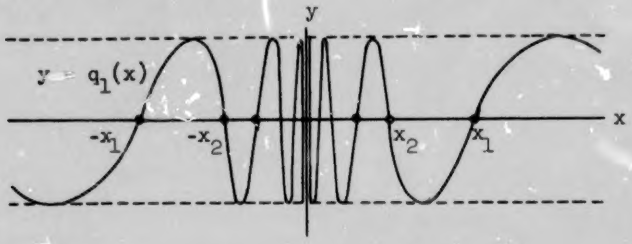
The range of f is the interval $[0,1]$.



(b) Let g be defined by

$$g(x) = \sin \frac{1}{x}, \; x \neq 0 .$$

The range of g is the interval $[-1,1]$.

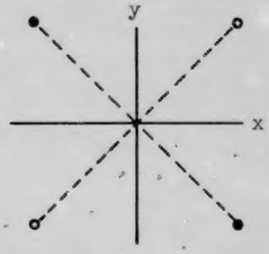


$g(x) = \sin \frac{1}{x}, \; x \neq 0$

(c) Let h be defined for $-1 \leq x \leq 1$ by

$$h(x) = \begin{cases} x, & x \text{ irrational}, \; -1 < x < 1 \\ -x, & x \text{ rational}, \; -1 \leq x \leq 1 . \end{cases}$$

The range is the interval $[-1,1]$.



2. On which of the following intervals is the function $f : x \longrightarrow \sin x$ increasing? decreasing? In each case locate the maximum and minimum values if any.

	Max.	Min.	Increasing	Decreasing
(a) $(-\frac{\pi}{2}, \pi)$	$f(\frac{\pi}{2})$	none		
(b) $[-\frac{\pi}{2}, \pi]$	$f(\frac{\pi}{2})$	$f(-\frac{\pi}{2})$		
(c) $[-\frac{\pi}{2}, 0]$	$f(0)$	$f(-\frac{\pi}{2})$	$[-\frac{\pi}{2}, 0]$	
(d) $(-\frac{\pi}{2}, \frac{\pi}{2})$	none	none	$(-\frac{\pi}{2}, \frac{\pi}{2})$	
(e) $[-\pi, \pi]$	$f(\frac{\pi}{2})$	$f(-\frac{\pi}{2})$		

A 3. Prove that $f : x \longrightarrow x^n$ is increasing for $x > 0$ and ranges over the positive reals (n, a natural number).

Note. The criterion n rational stated in the text is a misprint. Nonetheless, after the proof for a natural number n, a proof is appended for n rational and positive.

To prove f increasing, we use the First Principle of Mathematical Induction to prove the assertion,

$$A_n : \text{ if } 0 < a < b \text{ , then } a^n < b^n ,$$

is true for all n.

A_1 is true since $a^1 = a$ and $b^1 = b$. Assume that for $n = k$, if $0 < a < b$, then $a^k < b^k$ is true. Then, by Exercises A1-2, Number 6, we have

$$a \cdot a^k < b \cdot b^k$$

or

$$a^{k+1} < b^{k+1} .$$

Hence A_{k+1} is true and A_n is true for all n.

To prove that f ranges over the positive reals observe by the binomial theorem that

$$2^{mn} = (1 + 1)^{mn} \geq 1 + mn$$

(m, a natural number). Consequently,

$$f(2^m) \geq 1 + mn ,$$

and f is unbounded above. The function f extended to include 0 in its domain is continuous. Since $0^n = 0$, it follows by the Intermediate Value Theorem that the range of f contains all positive values.

Now we prove the result for a rational exponent $n = \frac{p}{q}$ where p and q are natural numbers. Since $g : x \longrightarrow x^q$ is increasing and continuous for $x > 0$, the inverse function $h : x \longrightarrow \sqrt[q]{x}$ is also increasing and continuous. Furthermore, the domain of g is the range of h and we see, therefore, that h ranges over the positive reals. Now the composition of continuous increasing functions is also continuous (see the solution of No.4), consequently the function $\phi h : x \longrightarrow x^{p/q}$ where $\phi : y \longrightarrow y^p$ for $y > 0$ is monotone and increasing. But the range of h is the domain of ϕ. Consequently, ϕh ranges over all positive numbers.

4. (a) Prove that $x \longrightarrow \sqrt{f(x)}$ is continuous and increasing whenever f is positive, continuous, and increasing.

This follows from the solution of Number 3 and the assertion, which we now prove, that the composition of continuous, increasing functions is continuous and increasing. We already have the continuity of a composition of continuous functions (Theorem 3-6e). The composition gf of increasing functions f and g is increasing because

$$a < b \Longrightarrow f(a) < f(b) \Longrightarrow g\big(f(a)\big) < gf(b) .$$

(b) Prove that $x \longrightarrow \sqrt[3]{f(x)}$ is continuous wherever f is continuous.

This follows from the solution of Number 3 and the remark of Number 4a.

5. Prove that $\lim\limits_{x \to 0} \dfrac{\sqrt{1 + x} - \sqrt{1 - x}}{x} = 1$.

Rationalize the numerator to obtain

$$\frac{\sqrt{1 + x} - \sqrt{1 - x}}{x} = \frac{2x}{x(\sqrt{1 + x} + \sqrt{1 - x})} ,$$

whence the result follows by the limit theorems.

6. (a) Assume that $f : x \longrightarrow \sin x$ is continuous, and prove that f is increasing. Hint: Use the identity

$$\sin y - \sin x = 2 \sin \left(\frac{y - x}{2}\right) \cos\left(\frac{y + x}{2}\right)$$

and consider as separate cases the closed intervals $[-\frac{\pi}{2}, 0]$ and $[0, \frac{\pi}{2}]$.

Note. The hint of the text is not used.

For x and y in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $y > x$, we have $0 < \frac{y - x}{2} \leq \frac{\pi}{2}$ and $-\frac{\pi}{2} < \frac{y + x}{2} < \frac{\pi}{2}$. Consequently, $\sin(\frac{y - x}{2})$ and $\cos(\frac{y + x}{2})$ are positive. Thus, for x and y in $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$$y > x \Longrightarrow \sin y - \sin x = 2 \sin\left(\frac{y - x}{2}\right) \cos\left(\frac{y + x}{2}\right) > 0 ,$$

which proves the result.

(b) Show that the range of f is an interval. Show that the domain of the inverse of $f : x \longrightarrow \sin x$ is an interval.

Since it is assumed that f is continuous, it follows by the Intermediate Value Theorem that its range is an interval. Since f is continuous so is its inverse, hence the range of the inverse is an interval.

7. Assume that the function $f : x \longrightarrow \tan x$ is continuous on the closed interval $[-\frac{\pi}{4}, \frac{\pi}{4}]$ and show that there is some number x where $-\frac{\pi}{4} < x < \frac{\pi}{4}$ such that $\tan x = \frac{\pi}{4}$. Can there be more than one such number?

Since $\tan(-\frac{\pi}{4}) = -1$ and $\tan \frac{\pi}{4} = 1$, we have

$$\tan(-\frac{\pi}{4}) < \frac{\pi}{4} < \tan \frac{\pi}{4} .$$

So, by the Intermediate Value Theorem, there must be a value u, $-\frac{\pi}{4} < u < \frac{\pi}{4}$, with $\tan u = \frac{\pi}{4}$. But $\tan x$ is increasing in $[-\frac{\pi}{4}, \frac{\pi}{4}]$, hence there is only one such number.

Λ 8. Prove that if f is continuous on the closed interval $[a,b]$ and all values of f are in $[a,b]$, then there is an x in $[a,b]$ for which $f(x) = x$.

Note. Problem really does not deserve "Λ."

Let $g(x) = x - f(x)$. Then g is continuous on $[a,b]$ since it is the sum of two continuous functions. If $f(a) = a$ or $f(b) = b$, the result is immediate.

Suppose $f(a) \neq a$ and $f(b) \neq b$. Then $g(a) = a - f(a) < 0$, since $a < f(a) \leq b$. Also, $g(b) = b - f(b) < 0$, since $a \leq f(b) < b$. Thus, by the Intermediate Value Theorem, since $g(a) < 0 < g(b)$, there must be a value u, $a < u < b$, with $g(u) = 0$. The result follows.

9. In which of the following intervals does $f : x \longrightarrow |x|$ have a maximum? a minimum? Justify your answer.

(a) $-1 < x < 1$ (b) $-1 \leq x < 1$ (c) $-1 < x \leq 1$ (d) $-1 \leq x \leq 1$

(a) f has a minimum but no maximum for $-1 < x < 1$. The minimum value is 0, since $0 \leq f(x) < 1$, for all x.

(b), (c), (d). The range of f is the closed interval $[0,1]$ hence f has a maximum and a minimum in each of these intervals.

10. Show that the equation $x^4 + x - 10 = 0$ has at least one solution between $x = -2$ and $x = -1$ and obtain an approximation to the solution within a tolerance of $\frac{1}{2}$.

Let $f(x) = x^4 + x - 10$. Then $f(-2) = 4$, and $f(-1) = -10$. Thus, by the Intermediate Value Theorem there is an x_0, $-2 < x_0 < -1$, such that $f(x_0) = 0$. Since $f(-\frac{3}{2}) = -10 + \frac{57}{16} < 0$, the root lies between -2 and $-\frac{3}{2}$.

11. Isolate each real root of the given equation by exhibiting an interval containing this root and no others. (Each equation has four roots.)

(a) $x^4 - x^3 - 9x^2 + 2x + 14 = 0$.

Let $f(x) = x^4 - x^3 - 9x^2 + 2x + 14$. Using synthetic division or direct substitution we obtain the following partial table of values.

x	-3	-2	-1	0	1	2	3	4
$f(x)$	35	-2	5	14	7	-10	-7	70

The equation has a root in each of the intervals $(-3, -2)$, $(-2, -1)$, $(1, 2)$, and $(3, 4)$.

(b) $2x^4 + 2x^3 - 3x^2 - x + 1 = 0$.

Let $f(x) = 2x^4 + 2x^3 - 3x^2 - x + 1$.

x	-2	-1	0	1	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{4}{3}$
$f(x)$	7	-1	1	1	$\frac{1}{8}$	$\frac{-7}{2048}$	$\frac{5}{128}$

The equation has a root in each of the intervals $(-2, -1)$, $(-1, 0)$, $(\frac{1}{2}, \frac{5}{8})$, and $(\frac{5}{8}, \frac{3}{4})$.

12. (a) Show that the equation $\cos^2 x = \sqrt{|x|}$ has at least one positive root x, where $x < \frac{\pi}{6}$.

Let $f(x) = \cos^2 x - \sqrt{|x|}$. f is continuous on the interval $[0, \frac{\pi}{6}]$. For the endpoints, $f(0) = 1$ and $f(\frac{\pi}{6}) = \frac{1}{4} - \sqrt{\frac{\pi}{6}} < 0$. Hence, by the Intermediate Value Theorem, there is some value u, $0 < u < \frac{\pi}{6}$, such that $f(u) = \cos^2 u - \sqrt{|u|} = 0$.

(b) Find the maximum and the minimum value of $f : x \longrightarrow \cos^2 x - \sqrt{|x|}$ on the closed interval $[0, \frac{\pi}{2}]$.

The function is decreasing on this interval. If $0 \leq x_1 < x_2 \leq \frac{\pi}{2}$, $0 \leq \sin x_1 < \sin x_2$ (No. 6), whence $-\sin^2 x_2 < -\sin^2 x_1$; also, $-\sqrt{x_2} < -\sqrt{x_1}$. Consequently,

$$1 - \sin^2 x_2 - \sqrt{x_2} < 1 - \sin^2 x_1 - \sqrt{x_1} ,$$

or

$$\cos^2 x_2 - \sqrt{x_2} < \cos^2 x_1 - \sqrt{x_1} ,$$

i.e.,

$$f(x_2) < f(x_1) .$$

Maximum: $f(0) = 1$. Minimum: $f(\frac{\pi}{2}) = -\sqrt{\frac{\pi}{2}}$.

∧ 13. Prove that if $p(x)$ is a polynomial of odd degree (with real coefficients), then the equation $p(x) = 0$ has a real root.

Note. Two solutions are provided. The alternative form is based on the Fundamental Theorem of Algebra (SMSG Elementary Functions, pp. 74, 75).

Take

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1} x^{n+1} + \dots + a_0 \\ &= x^n [a_n + \frac{a_{n-1}}{x} + \dots + \frac{a_0}{x^n}] , \end{aligned}$$

where $a_n \neq 0$. For $|x| > 1$ we have

$$\begin{aligned} \epsilon(x) &= \left| \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n} \right| \\ &\leq \frac{1}{|x|} \left(|a_{n-1}| + |a_{n-2}| + \dots + |a_0| \right) \\ &\leq \frac{A}{|x|} \quad \text{where } A \text{ is a constant.} \end{aligned}$$

For $|x| = k > \frac{A}{a_n}$ we have $|a_n| > \epsilon(x)$. For $x = \pm K$, then, $p(x)$ assumes opposite signs. It follows by the Intermediate Value Theorem that p has a zero in $[-K,K]$.

Alternatively:

$$p(a + ib) = u(a,b) + iv(a,b)$$

where u and v are real functions of a and b (by using the binomial theorem). If $a + ib$ is a root, then

$$u(a,b) = v(a,b) = 0 .$$

Also, $p(a - ib) = u(a,b) - iv(a,b)$ and so is $a - ib$ a root. Thus, all complex roots of a polynomial with real coefficients must occur in pairs. Consequently, since there is an odd number of roots, there is at least one real one.

14. Prove that the equation $x^n = a$ has exactly one negative root if n is an odd positive integer and $a < 0$.

From Number 13 we know that x^n has a root. For $x \geq 0$, $x^n \geq 0$. Consequently, any root must be negative. Since x^n is increasing for n odd, it follows that $f : x$———x^n has an inverse and the root is unique.

The proof that f is increasing goes by induction. For $n = 1$, we have, trivially, $0 < a < b \Rightarrow 0 < a < b$. Suppose $0 \leq a < b \Rightarrow 0 \leq a^n < b^n$. Then if $0 < a < b$,

$$a \cdot a^n \leq b \cdot a^n < b \cdot b^n ,$$

whence,

$$a^{n+1} < b^{n+1} .$$

Now if n is odd, $a < b$, we consider three cases.

Case 1. $0 \leq a$. Theorem already proved.

Case 2. $a < 0 \leq b$. Then, since n is odd, $a^n < 0 \leq b^n$.

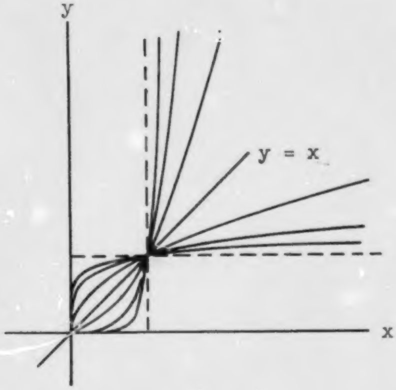
Case 3. $b < 0$. Then, $0 < |b| < |a|$, whence,

$$0 < |b|^n < |a|^n .$$

But, since n odd, $|a|^n = -a^n$ and $|b|^n = -b^b$; therefore,

$$a^n < b^n < 0 .$$

15. Sketch the curves $y = x^n$ and $y = x^{1/n}$, $n = 0 , 1 , 2 , \ldots ,$ using the same set of axes.



16. Prove that if f is continuous and has an inverse on an interval, then f is strongly monotone on the interval.

Note. This problem should appeal to the "better" students. You may wish to peak it.

Let a and b be any two points of the domain of f with $a < b$. Since f has an inverse we have $f(a) \neq f(b)$. Suppose $f(a) < f(b)$ (the proof for $f(a) > f(b)$ is entirely similar). We show that f is increasing on $[a,b]$. If not, there exist points u , v in $[a,b]$ for which $u < v$ and $f(u) \geq f(v)$. The proof now utilizes exactly that part of the argument in Theorem 5-2b in which it is established that there exist two points c_1 and c_2 in $[a,b]$ such that $f(c_1) = f(c_2)$. From $f(c_1) = f(c_2)$, we see at once that the function f cannot be one-to-one.

We conclude that f must be strongly monotone on every subinterval. It follows that f must be increasing everywhere, for if $\alpha < \beta$ then f is strongly monotone on the interval $[A,B]$ where $A = \min\{a,\alpha\}$, $B = \max\{b,\beta\}$. Since $[A,B]$ contains $[a,b]$ where f is increasing, it follows that f is increasing on $[A,B]$. Since $[A,B]$ contains $[\alpha,\beta]$ it follows that $f(\alpha) < f(\beta)$.

17. The temperature at any point of a thin circular ring is a continuous function of the point's position. Show that there is a pair of antipodes (points at opposite ends of a diameter) having the same temperature.

We have $T : \theta \longrightarrow T(\theta)$, $0 \leq \theta \leq 2\pi$, T continuous. We are to show that $T(\theta_0) = T(\theta_0 + \pi)$ for some $\theta_0 \leq \pi$.

Define K by

$$K(\theta) = T(\theta + \pi) - T(\theta) \ , \ 0 \leq \theta \leq \pi \ .$$

We want to show that $K(\theta) = 0$ for some θ , $0 < \theta < \pi$.

$$\begin{aligned} K(0) &= T(\pi) - T(0) \\ K(\pi) &= T(2\pi) - T(\pi) \\ &= T(0) - T(\pi) \\ &= -K(0) \ . \end{aligned}$$

Since T is continuous, K is continuous, and by the Intermediate Value Theorem, $T(\theta) = 0$ for some θ_0 , $0 < \theta_0 < \pi$.

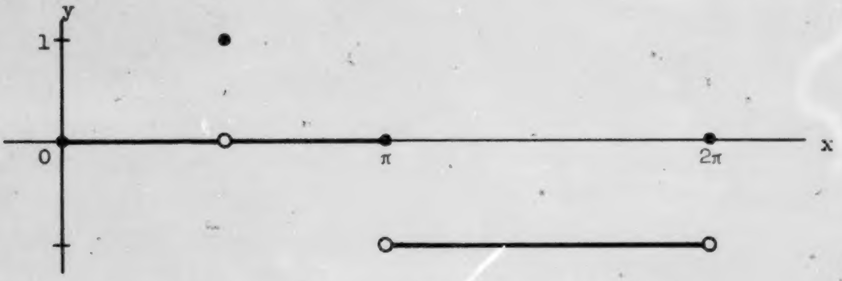
18. Sketch the graphs and determine how many points of discontinuity there are in the interval $[0,2\pi]$ of the following functions:

(a) $f : x \longrightarrow [\sin x]$

(b) $f : x \longrightarrow [2 \sin x]$

(c) $f : x \longrightarrow [a \sin x]$

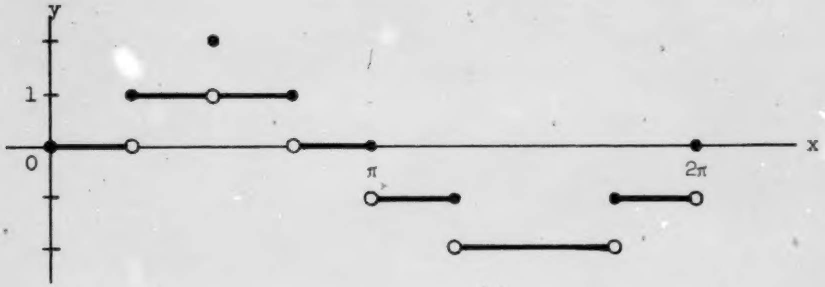
(a)



$f : x \longrightarrow [\sin x]$

There are 3 points of discontinuity.

(b)



$f : x \longrightarrow [2 \sin x]$

There are 7 points of discontinuity.

(c) If a is a positive integer, an increase of 1 in a increases the number of discontinuities by 4 . Since reflection of the graph in the x-axis does not change the number of discontinuities, if a is an integer f has $4|a| - 1$ discontinuities in the interval $[0,2\pi]$.

If a is not an integer, the number of discontinuities is decreased by 1 giving $4|a| - 2$ discontinuities.

The two cases are combined to give the following result: for any real number a , f has $4[|a|] - 1$ discontinuities in $[0,2\pi]$.

A19. If f is periodic with periods 1 and $\sqrt{2}$ (i.e., $f(x) = f(x + 1)$ and $f(x) = f(x + \sqrt{2})$ for all x) and is continuous at least at one point, show that f must be constant.

Since 1 and $\sqrt{2}$ are periods and any sum or difference of periods is also a period, it follows that every number of the form

$$\text{(a)} \qquad p = m + n\sqrt{2} ,$$

where m and n are integers, is a period. Now let a be the assumed point of continuity of f . We show for any positive δ that there exists a period p of the form (a) such that $x + p$ lies within a δ-neighborhood of a . Consequently, by the definition of continuity, given any positive ϵ we can guarantee

$$\text{(b)} \qquad |f(x) - f(a)| = |f(x + p) - f(a)| < \epsilon .$$

Thus $f(x) = f(a)$, for any x , which proves the constancy of x .

First we shall show that there exists a period $\sigma = \mu + \nu\sqrt{2}$ such that

$$\text{(c)} \qquad 0 < \sigma < \delta .$$

For $\lambda = \left[\frac{x - a}{\sigma}\right]$, we will then have

$$\lambda\sigma \leq x - a < (\lambda + 1)\sigma ,$$

whence

$$0 \leq (x - \lambda\sigma) - a < \sigma < \delta .$$

Consequently, with $p = \lambda\sigma$ in (b), the argument will be complete.

Now we prove the result of (c). First we give a proof specifically applicable to this problem. Observe, since $0 < (\sqrt{2} - 1) < \frac{1}{2}$, that $0 < (\sqrt{2} - 1)^n < \frac{1}{2^n}$. But we can write

$$(\sqrt{2} - 1)^n = Q_n\sqrt{2} - P_n$$

where Q_n and P_n are integers (in fact, these are related to the natural numbers p_n and q_n of Exercises A3-1, No. 18 by $p_n = |P_n|$, $q_n = |Q_n|$.) We choose n so large that $\frac{1}{2^n} < \delta$, hence,

$$\sigma = Q_n\sqrt{2} - P_n < \delta .$$

The following argument is general (given any two incommensurable periods α and β , we can find $0 < \mu\alpha + \nu\beta < \epsilon$), but it relies on A1-5. Let

τ be the greatest lower bound of the positive periods given by (a). If $\tau > 0$ then τ is a period of the form (a). For if τ were not such a period then the interval $(\tau, 2\tau)$ would have to contain a period β_1 of the form (a) and, likewise, the interval (τ, β_1) would have to contain a period β_2 of the form (a). Thus $0 < \beta_1 - \beta_2 < \tau$. But $\beta_1 - \beta_2$ must then be a positive period of the form (a), but less than τ, contrary to the definition of τ as a lower bound.

If τ. is a period, then any other period of the form (a) must have the form $k\tau$ where α is an integer. For if β is any period we take

$$k = \left[\frac{\beta}{\tau}\right],$$

from which

$$0 \leq \beta - k\tau < \tau .$$

Since $\beta - k\tau$ is a period of the form (a) it cannot be positive, for then τ could not be a lower bound. It follows that $\beta - k\tau = 0$.

From the last result, it follows that $1 = k_1\tau$ and $\sqrt{2} = k_2\tau$ where k_1 and k_2 are positive integers. Consequently, $\sqrt{2} = \frac{k_1}{k_2}$ contradicting our knowledge that $\sqrt{2}$ is irrational.

We could conclude that the hypothesis $\tau > 0$ is not valid; we can only have $\tau = 0$. But 0 is not a positive period of the form (a). Since 0 is the greatest lower bound of such periods, we conclude that there exists at least one such period in the interval $(0, \delta)$.

The greatest lower bound of the positive periods of a function is called the fundamental period. In the manner used to prove the result above, we can establish:

1. If the fundamental period is positive, then all periods are multiples of the fundamental period.
2. If the function has two incommensurable periods, then the fundamental period is zero.
3. If the fundamental period is zero and the function is continuous at any point, then the function is constant.

Λ 20. If g is continuous with $g(0) = g(1) = 1$ and, in the interval $[0,1]$, $g(x^2) = \left(g(x)\right)^2$, show that $g(x) = 1$ in $[0,1]$.

Note. Although identified with Λ, this problem is well within the reach of most students who have "survived" Chapter 3.

Iterate to obtain

(a) $$g(x^{2^n}) = g(x)^{2^n} .$$

For $x < 1$, x^{2^n} approximates 0 within any given tolerance for n sufficiently large. By the continuity of g at 0 it follows that $g(x^{2^n}) = g(x)^{2^n}$ approximates $g(0)$, similarly. However, if $g(x) > 1$ then $g(x)^{2^n}$ will be larger than any given number for n large enough, and if $g(x) < 1$ then $g(x)^{2^n}$ will be within any given tolerance of 0 for n large enough; either condition contradicts the condition that g is continuous at 0 and $g(0) = 1$.

21. The real roots of the equation $x^n + ax + b = 0$ (n a positive integer) can be "determined" by finding the intersections of the curves

$$y = x^n \quad \text{and} \quad y = -ax - b .$$

Verify the following table for the number of real roots of $x^n + ax + b = 0$.

(a) If n is even, and $\begin{cases} b > 0 \text{ , there are two or none,} \\ b < 0 \text{ , there are two.} \end{cases}$

(b) If n is odd, and $\begin{cases} a > 0 \text{ , there is one,} \\ a < 0 \text{ , there are three or one.} \end{cases}$

Give numerical examples to illustrate each of the four cases.

Take $f : x \longrightarrow x^n + ax + b$. First determine the parts of the domain in which f is monotone. Suppose $x < y$.

$$\begin{aligned} f(y) - f(x) &= y^n - x^n + a(y - x) \\ &= (y - x)[(y^{n-1} + y^{n-2}x + \dots + x^{n-1}) + a] , \\ &= (y - x)[\phi(x,y) + a] . \end{aligned}$$

Thus,

(a) $$\operatorname{sgn}[f(y) = f(x)] = \operatorname{sgn}[\phi(x,y) + a] .$$

If $x \geq 0$, then $\phi(x,y) > nx^{n-1}$; hence

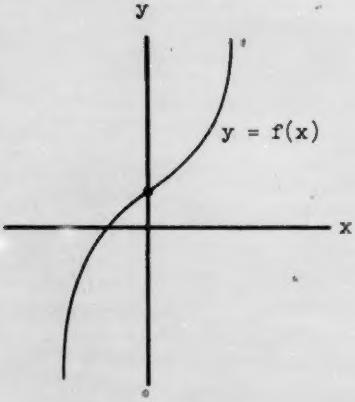
$$\text{(b)} \qquad x^{n-1} \geq -\frac{a}{n} \Longrightarrow f(y) > f(x), \qquad (x \geq 0).$$

Case 1. n odd.

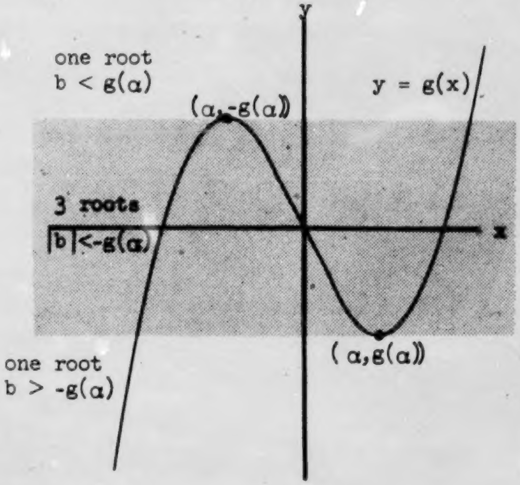
For n odd, the graph $y = f(x)$ has a center of symmetry at $(0,b)$; i.e., the function $x \to f(x) - b = x^n + ax$ is odd. The properties of f everywhere are then determined by the properties for $x \geq 0$. Now, if $a \geq 0$, condition (b) always holds since $n - 1$ is even. Thus, f is increasing for $x \geq 0$, and by symmetry, also for $x < 0$. Since f takes on negative and positive values (for large negative and positive x) it follows that f has a root (Intermediate Value Theorem) and since f is increasing, therefore one-to-one, the root is unique. If $a < 0$, then condition (b) holds for $x \geq \sqrt[n-1]{-\frac{a}{n}}$. Furthermore,

$$\text{(c)} \qquad y^{n-1} \leq -\frac{a}{n} \Longrightarrow \phi(x,y) + a < 0, \qquad (0 \leq x < y).$$

From (a) it follows that f is decreasing on $(0,\alpha)$ where $\alpha = \sqrt[n-1]{-\frac{a}{n}}$. Again by symmetry, f is increasing for $x \leq -\alpha$ and $x \geq \alpha$ and decreasing for $-\alpha \leq x \leq \alpha$. Since the slope of the curve for every value of b is the same for a given a, we can analyze the situation by referring to the graph of $g : x \to x^n + ax$ and count the roots by considering intersections with horizontal lines. Thus there are three roots or one depending on whether a horizontal line crosses the decreasing segment of the graph or not. (The special case of two roots occurs at the endpoints of the segment.)



n odd, a ≥ 0



n odd, a < 0

Case 2. n even.

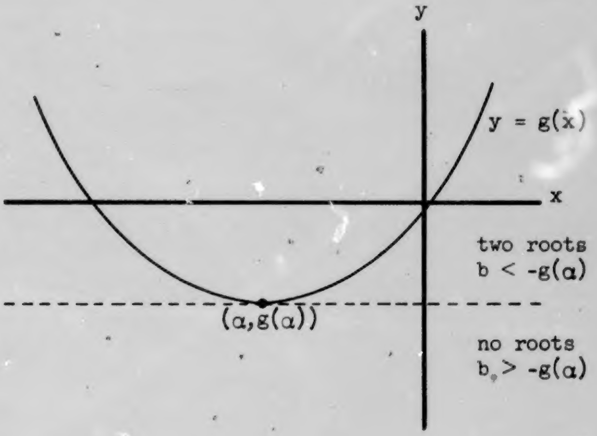
For n even, $n - 1$ odd, conditions (b) and (c) hold as before. Without symmetry, it is necessary to consider f on the negative part of its domain as well. For $x < y \le 0$, set $u = -x$, $v = -y$. Then

$$\phi(x,y) = (-1)^{n-1}\,\phi(u,v) = -\phi(u,v) .$$

From $u > v$ it follows that $nv^{n-1} < \phi(u,v) < nu^{n-1}$; hence,

$$nx^{n-1} < \phi(x,y) < ny^{n-1}$$

and conditions (b) and (c) hold for the entire domain of f. Thus f decreases for $x \le \alpha$ and increases for $x \ge \alpha$. The situation is depicted below.



n even

Teacher's Commentary

Chapter 4

DIFFERENTIATION

The proofs of theorems for the derivatives of algebraic functions and circular functions are given in this chapter. Several manipulative exercises have been included in order to establish a need for more efficient methods. To accomplish this end, care should be taken to avoid premature application of theorems.

TC4-1. Introduction.

In this section we assert the equality of the numbers

$$\lim_{z \to x} \frac{f(z) - f(x)}{z - x} \quad \text{and} \quad \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

where $z = x + h$. This equality can be established analytically by application of Theorem 3-6e. Let

$$r : z \longrightarrow \begin{cases} \dfrac{f(z) - f(x)}{z - x} & , \ z \neq x \\ f'(x) & , \ z = x \end{cases}$$

and $g : h \longrightarrow x + h$; then

$$rg : h \longrightarrow \begin{cases} \dfrac{fg(h) - f(x)}{g(h) - x} & , \ h \neq 0 \\ f'(x) & , \ h = 0 . \end{cases}$$

If both r and g are continuous then rg is continuous and $\lim_{h \to 0} rg(h) = f'(x)$.

Solutions Exercises 4-1

1. Using the definition of the derivative, find $f'(x)$, where $f(x)$ equals:

   (a) $f(x) = 2x^2 - x + 4$

$$f'(x) = \lim_{h \to 0} \frac{2(x + h)^2 - (x + h) + 4 - 2x^2 + x - 4}{h}$$

$$= \lim_{h \to 0} \frac{4xh + 2h^2 - h}{h} = \lim_{h \to 0} 4x + 2h - 1 = 4x - 1$$

(b) $f(x) = 1 - x^3$

$$f'(x) = \lim_{h \to 0} \frac{1 - (x + h)^3 - (1 - x^3)}{h}$$

$$= \lim_{h \to 0} \frac{-3x^2h - 3xh^2 - h^3}{h} = \lim_{h \to 0} (-3x^2 - 3xh - h^2)$$

$$= -3x^2$$

(c) $f(x) = \frac{1}{\sqrt{x}}$

$$f'(x) = \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{\sqrt{x + h}} - \frac{1}{\sqrt{x}}\right)$$

$$= \lim_{h \to 0} \frac{(\sqrt{x} - \sqrt{x + h})(\sqrt{x} + \sqrt{x + h})}{(h\sqrt{x + h}\ \sqrt{x})(\sqrt{x} + \sqrt{x + h})}$$

$$= \lim_{h \to 0} \frac{-h}{h(x\sqrt{x + h} + (x + h)\sqrt{x})} = \lim_{h \to 0} \frac{-1}{x\sqrt{x + h} + (x + h)\sqrt{x}}$$

$$= \frac{-1}{2x\sqrt{x}} = \frac{-1}{2x^{3/2}}$$

(d) $f(x) = \frac{1}{x^2}$

$$f'(x) = \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{(x + h)^2} - \frac{1}{x^2}\right)$$

$$= \lim_{h \to 0} \frac{x^2 - (x + h)^2}{h(x + h)^2x^2} = \lim_{h \to 0} \frac{-2xh - h^2}{h(x + h)^2x^2}$$

$$= \frac{-2x}{x^4} = \frac{-2}{x^3}$$

(e) $f(x) = x^3 - 3x + 4$

$$f'(x) = \lim_{h \to 0} \frac{(x + h)^3 - 3(x + h) + 4 - (x^3 - 3x + 4)}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h}$$

$$= 3x^2 - 3$$

(f) $f(x) = \frac{1}{ax + b}$ $(a , b$ constant, $x \neq -\frac{b}{a} , a \neq 0)$

$$f'(x) = \lim_{h\to 0} \frac{1}{h} \left(\frac{1}{a(x + h) + b} - \frac{1}{ax + b}\right)$$

$$= \lim_{h\to 0} \frac{(ax + b) - (ax + ah + b)}{h(ax + ah + b)(ax + b)}$$

$$= \frac{-a}{(ax + b)^2}$$

(g) $f(x) = ax^2 + bx + c$ $(a , b , c$ constants)

$$f'(x) = \lim_{h\to 0} \frac{a(x + h)^2 + b(x + h) + c - ax^2 - bx - c}{h}$$

$$= \lim_{h\to 0} \frac{2axh + ah^2 + bh}{h}$$

$$= \lim_{h\to 0} (2ax + ah + b) = 2ax + b$$

(h) If $x > 1$, $f(x) = x - 1$
and $f'(x) = 1$.

If $x < 1$, $f(x) = 1 - x$
and $f'(x) = -1$.

Thus, $f'(x) = \frac{|x - 1|}{x - 1} = \operatorname{sgn}(x - 1)$, $x \neq 1$.

(Derivative does not exist at $x = 1$.)

(i) $f(x) = ax + \frac{b}{x}$

$$f'(x) = \lim_{h\to 0} \frac{1}{h} \left(a(x + h) + \frac{b}{x + h} - ax - \frac{b}{x}\right)$$

$$= \lim_{h\to 0} \frac{1}{h} \left(ah - \frac{bh}{(x + h)x}\right) = \lim_{h\to 0} \left(a - \frac{b}{(x + h)x}\right)$$

$$= a - \frac{b}{x^2}$$

2. Given the line $y = 3x + 2$, find [illegible] slope at the points $(0,2)$, $(-2,-4)$, and $(2,8)$.

$$f'(a) = \lim_{x \to a} \frac{(3x + 2) - (3a + 2)}{x - a} = 3$$

$f'(0) = 3$

$f'(-2) = 3$

$f'(2) = 3$

3. If $f(x) = 1 + 2x - x^2$, find the slope of the graph of f at points corresponding to

$$f'(a) = \lim_{x \to a} \frac{1 + 2x - x^2 - 1 - 2a + a^2}{x - a}$$

$$= \lim_{x \to a} (2 - x - a) = 2 - 2a$$

(a) If $x = 0$ $f'(0) = 2$.

(b) If $x = \frac{1}{2}$ $f'(\frac{1}{2}) = 1$.

(c) If $x = 1$ $f'(1) = 0$.

(d) If $x = -10$ $f'(-10) = -18$.

4. If $f(x) = x^3 + 2x + 1$, find all x such that

(a) $f'(x) = 0$

(b) $f'(x) = -1$

(c) $f'(x) = 4$

(d) $f'(x) = 20$

$f'(x) = 3x^2 + 2 = n$ whence $x = \pm \sqrt{\frac{n - 2}{3}}$.

Thus, n must be ≥ 2.

(a), (b) there are no values.

(c) $x = \pm \frac{\sqrt{6}}{3}$

(d) $x = \pm \sqrt{6}$

TC4-2. Rational Operations.

In Section 3-4 (Corollary to Theorems 3-4a and b), we considered the class of linear combinations: here we prove the linearity of differentiation. Thereafter, the theorem on differentiation of products is extended to include integral powers of differentiable functions. This sequence has the advantage of providing theorems on differentiating polynomial and rational functions (of any functions whose derivatives we know) before the chain rule is established. The theorems of Section 4-2 enable us to differentiate a large class of functions. Derivatives of fractional powers and composite functions (in general) are considered in later sections.

Exercises have been freely interspersed throughout this section. These exercises have been designed to give the student immediate application of theorems as they are developed and, successively, to establish the need for the development of more expedient methods of differentiation. It should be noted that the placement of sets of exercises, in no way, is intended to indicate a time block of work; nor does the arrangement within a set indicate the difficulty of individual items. A sufficient number and variety have been included to provide for the needs of each student.

Solutions to exercises are not spelled out in detail except when an exercise has been designed to develop a particular skill; in such cases, solutions may be given in detail. Quite often a considerable amount of algebraic manipulation may be required to arrive at a simplified answer. A student's algebraic dexterity should improve as he attempts to submit answers in a form you consider suitable.

Solutions Exercises 4-2a

1. Evaluate

(a) $D(4|x| + 6\sqrt{x}) = 4 + \frac{3}{\sqrt{x}}, \ x > 0$.

(b) $D(5x^2 + \frac{2}{x}) = 10x - \frac{2}{x^2}$

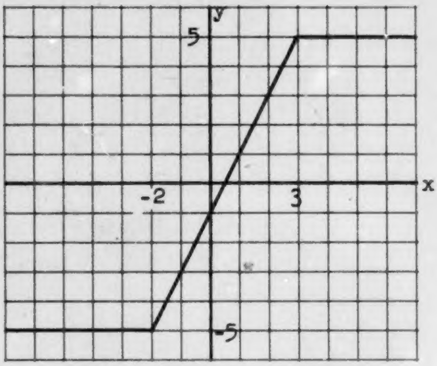
(c) $$D\left(|7x| + \frac{3}{2x-1}\right) = 7\,\frac{|x|}{x} - \frac{6}{(2x-1)^2}$$

$$= 7 \operatorname{sgn} x - \frac{6}{(2x-1)^2}\,,\quad x \neq 0\,.$$

(d) $D(|ax| - |bx|) = (a + b) \operatorname{sgn} x\,,\ a > 0\,,\ b < 0\,,\ x \neq 0\,.$

2. Consider $g : x \longrightarrow |x + 2| - |3 - x|$.

(a) Sketch the graph of g .



(b) Define $g(x)$ explicitly in terms of linear functions for all real x .

If $x < -2$, then $x + 2 < 0$ and $3 - x > 0$.
If $-2 \leq x \leq 3$, then $x + 2 \geq 0$ and $3 - x \geq 0$.
If $x > 3$, then $x + 2 > 0$ and $3 - x < 0$.
$g(x) = -(x + 2) - (3 - x) = -5$, if $x < -2$
$g(x) = (x + 2) - (3 - x) = 2x - 1$, if $-2 \leq x \leq 3$
$g(x) = (x + 2) - (-3 + x) = 5$, if $x > 3$

(c) For what values of x is the derivative not defined?

$f'(x)$ is not defined at $x = -2$ and at $x = 3$.

3. Consider $f : x \longrightarrow [x]$ ($[x]$ is the integer part of x, defined in A2)

   (a) Find $f'(x)$ if it exists, at each of the values $x = -2.8$, $x = 0.6$, $x = 2$.

   $f'(-2.8) = 0$, $f'(0.6) = 0$ but

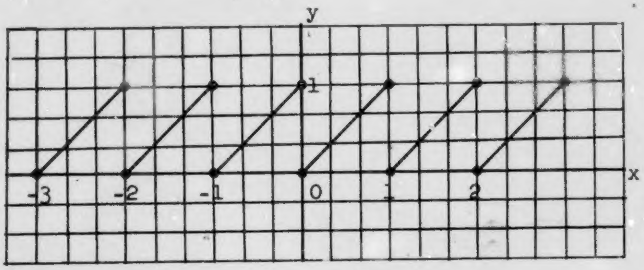
   $f'(x)$ is not defined at $x = 2$.

   (b) Find the domain of the derivative f'.

   The domain of f' is the set of all non-integral real numbers since f' is not defined at $x = n$, n an integer.

4. Consider $f : x \longrightarrow x - [x]$.

   (a) Draw the graph of f.



   (b) Find $f'(-1.5)$ and $f'(2.3)$ and describe the domain of the derivative.

   The domain of f' is the set of all non-integral real numbers.

   $f'(-1.5) = 1$, $f'(2.3) = 1$.

5. Extend Theorem 4-2a to a general linear combination of functions

$$\phi : x \longrightarrow c_1f_1(x) + c_2f_2(x) + \dots + c_nf_n(x) .$$

The proof of the extension of Theorem 4-2a by mathematical induction is given. Theorem 4-2a serves as verification that the theorem holds for $n = 2$.

Assume that the theorem is true for same positive integer $n = k$,

$$D_x[c_1f_1(x) + c_2f_2(x) + \dots + c_kf_k(x)] =$$
$$c_1f_1'(x) + c_2f_2'(x) + \dots + c_kf_k'(x)$$

Then
$$D_x[c_1f_1(x) + c_2f_2(x) + \dots + c_{k+1}f_{k+1}(x)] =$$
$$D_x[c_1f_1(x) + \dots + c_kf_k(x)] + D_xc_{k+1}f_{k+1}(x) =$$
$$c_1f_1'(x) + \dots + c_kf_k'(x) + c_{k+1}f_{k+1}'(x)$$

If the theorem holds for $n = k$, it holds for $n = k + 1$, hence the theorem holds for all n .

6. For each of the following functions, find the derivative and describe the domain of the derivative.

(a) $f : x \longrightarrow |x^2 - 2|$

$f'(x) = 2x \operatorname{sgn}(x^2 - 2) , x^2 \neq 2$.

(b) $f : x \longrightarrow [2x^2]$

$f'(x) = 0 , x^2 \neq \frac{n}{2}$, n an integer.

(c) $f : x \longrightarrow |[x + 1]|$

$f'(x) = 0 , x \neq n$, n an integer.

(d) $f : x \longrightarrow [|x + 1|]$.

$f'(x) = 0 , x \neq n$, n a non-zero integer.

(e) $f : x \longrightarrow \operatorname{sgn}(1 - \sqrt[3]{x})$.

$f'(x) = 0 , x \neq 1$.

(f) $f : x \longrightarrow \max\{x^3 , 4|x|\}$.

For $x > 2$, $f'(x) = 3x^2$,
for $x < 2$, $f'(x) = 4 \operatorname{sgn} x$, $x \neq 0$.

(g) $f : x \longrightarrow \min\{[x] , \max\{x^3,2x^2\}\}$.

$f'(x) = 0$, x not an integer.

(h) $f : x \longrightarrow \operatorname{sgn} (\min\{x^3 - 1 , 7\})$.

$f'(x) = 0$, $x = 1$.

7. Right-hand and left-hand derivatives are defined in terms of right-hand and left-hand limits (see Exercises 3-4, No. 16) as follows:

Right-hand derivative: $$D^+f(x) = \lim_{h \sim 0^+} \frac{f(x + h) - f(x)}{h}$$

Left-hand derivative: $$D^-f(x) = \lim_{h \sim 0^-} \frac{f(x + h) - f(x)}{h}$$

In particular, $D^+|x| = D^-|x| = 1$, $x > 0$,
$D^+|x| = D^-|x| = -1$, $x < 0$,
$D^+|x| = 1 = -D^-|x|$, $x = 0$.

(a) Show that $D^-|x^5| = D^+|x^5|$ for all x .

For $x > 0$, $D^+|x^5| = 5x^4 = D^-|x^5|$;
for $x < 0$, $D^+|x^5| = -5x^4 = D^-|x^5|$; and
for $x = 0$, $D^+|x^5| = 0 = D^-|x^5|$.

(b) For what values of x does $D^-|x^3 - 2| = D^+|x^3 - 2|$?

$D^-|x^3 - 2| = D^+|x^3 - 2|$ for all x , $x \neq \sqrt[3]{2}$.

(c) Show that a function is differentiable at a point if and only if it has equal right-hand and left-hand derivatives at the point.

A function is differentiable at a point if and only if the limit of the difference quotient exists at that point. Therefore

$$\lim_{|h|\to 0} \frac{f(x+h) - f(x)}{h}$$ must exist, that is

$$\lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0^-} \frac{f(x+h) - f(x)}{h}$$ and conversely.

Solutions Exercises 4-2b

1. Find the derivatives of the following functions.

(a) $Dx(2x - 3) = 4x - 3$

(b) $D(4x - 2)(4 - 2x) = -16x + 20$

(c) $D(x^2 + x + 1)(x^2 - x + 1) = 4x^3 + 2x$

(d) $D\sqrt{x}\,(ax + b)^3 = \dfrac{(ax + b)^3}{2\sqrt{x}} + 3a\sqrt{x}\,(ax + b)^2$

(e) $D(\frac{1}{x} \cdot \sqrt{x}) = -\frac{1}{2}x^{-3/2}$

(f) $D\left(\frac{1}{x}(5x + 2)\right) = -\dfrac{2}{x^2}$

(g) $D(|x| \cdot \frac{1}{x^2}) = \dfrac{\operatorname{sgn} x}{x^2}$, $x \neq 0$

(h) $Dx^{7/2} = \frac{7}{2}x^{5/2}$

(i) $D(3x^4 + \frac{1}{\sqrt{x}}) = 12x^3 - \frac{1}{2}x^{-3/2}$

(j) $D\left(3x^2(x^2 - 5)\right) = 12x^3 - 30x$

(k) $D|13x^2 - 36 - x^4| = (4x^3 - 26x) \cdot \operatorname{sgn}(x^4 - 13x^2 + 36)$, $|x| \neq 2$, 3

(ℓ) $D|x^4 + 5x^2 - 36| = (4x^3 + 10x) . \operatorname{sgn}(x^4 + 5x^2 - 36)$, $|x| \neq 2$

2. **Evaluate**

(a) $D(3x^2 + 5x - 1)^2 = 2(3x^2 + 5x - 1)(6x + 5)$

(b) $D(3 - 5x)^3 = -15(3 - 5x)^2$

(c) $D(3 - 5x)^4 = -20(3 - 5x)^3$

(d) $D\, x(\sqrt{x} - 1)^2 = 2x - 3\sqrt{x} + 1$

(e) $D(x + \frac{1}{x})^2 = 2x - \frac{2}{x^3}$

(f) $D\left(\frac{x^{3/2}}{3} - \frac{x^{1/2}}{2} + x^{-1/2}\right) = \frac{\sqrt{x}}{2} - \frac{1}{4\sqrt{x}} - \frac{1}{2x\sqrt{x}}$

(g) $D\left(4\sqrt{x^3} - 2\sqrt{x} + \frac{1}{\sqrt{x}}\right) = 6\sqrt{x} - \frac{1}{\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$

(h) $D\left([x^2 - 1](x^2 - 3x + 1)\right) = [x^2 - 1](2x - 3)$ for $x^2 \neq n$,
n an integer.

(i) $D\left(x([x + 1] - 1)^n\right) = ([x + 1] - 1)^n\, Dx + xD([x + 1] - 1)^n$
$= ([x + 1] - 1)^n$, for x not an integer.

3. **Prove the corollary to Theorem 4-2b.**

If f' exists and if $F(x) = [f(x)]^2$, then

$$\begin{aligned} F'(x) &= 2[f(x)]\, Df(x) \\ &= 2f(x) \cdot f'(x) \,. \end{aligned}$$

Proof.

$$\begin{aligned} &D\big(f(x) \cdot f(x)\big) \\ &= f(x)\, Df(x) + f(x)\, Df(x) \\ &= 2f(x)\, Df(x) \\ &= 2f(x) \cdot f'(x) \,. \end{aligned}$$

4. Find the listed derivative by two methods: first, expand and then differentiate; second, use the product formula.

(a) $D(x^2 + 1)^2 = 4x^3 + 4x$

(b) $D\, x^2(x^2 + 1)^2 = 2x(3x^4 + 4x^2 + 1)$

(c) $D(x + 1)(x^2 - x + 1) = 3x^2$

(d) $D\big(adx^4 + (ae + bd)x^3 + (af + be + dc)x^2 + (bf + ce)x + cf\big)$

$= 4adx^3 + 3x^2(ae + bd) + 2x(af + be + dc) + bf + ce$

5. Find the derivative of each of the following functions in as many ways as you can and describe its domain. (Do not overlook the definition of the derivative.)

	Domain
(a) $D(x\lvert x\rvert) = 2\lvert x\rvert$,	all x .
(b) $D\lvert x\rvert^2 = 2x$,	all x .
(c) $Dx[x] = [x]$,	x not an integer.
(d) $D[x]^2 = 0$,	x not an integer..
(e) $D\lvert x\rvert[x] = \dfrac{x[x]}{\lvert x\rvert}$,	x not an integer.
(f) $D\lvert x\rvert[x]^2 = \dfrac{x[x]^2}{\lvert x\rvert}$,	x not an integer.

(g) $Dx[4 - x^2] = [4 - x^2]$, $x \neq \pm\sqrt{|4 - n|}$, n an integer.

(h) $$Dx \max\{x, 2 - x^2\} = \begin{cases} 2 - 3x^2 & \text{for } -2 < x < 1 \\ 2x & \text{for } x > 1 \text{ or } x < -2 \end{cases}$$

(i) $D(x[x]|x|) = 2|x|[x]$, x not an integer.

(j) $D[|x|] \cdot |[x]| = 0$, x not a positive or negative integer.

Solutions Exercises 4-2c

1. (a) Prove Theorem 4-2c.

If f' exists and if $F(x) = [f(x)]^n$, then

$$F'(x) = n[f(x)]^{n-1} f'(x)$$

for any positive integer n .

Proof. If $n = 2$, $F'(x) = 2 f(x)f'(x)$.

Suppose for $n = k$, $F'(x) = k[f(x)]^{k-1} f'(x)$, then for

$G(x) = [f(x)]^{k+1}$

$$\begin{aligned} G'(x) &= f(x)D[f(x)]^k + [f(x)]^k D\big(f(x)\big) \\ &= f(x)\big(k[f(x)]^{k-1} f'(x)\big) + [f(x)]^k f'(x) \\ &= (k + 1)[f(x)]^k f'(x) \quad . \end{aligned}$$

(b) Prove Corollary to Theorem 4-2c.

If $G(x) = x^n$, then $G'(x) = nx^{n-1}$ for any positive integer n .

Proof. Let $f(x) = x$, then $f'(x) = 1$.

By Theorem 4-2c, $D_x[f(x)]^n = D_x[x^n] = n[f(x)]^{n-1}f'(x) = nx^{n-1}$.

(c) Prove Corollaries 2 and 3 to Theorem 4-2c.

Corollary 2. A polynomial function p, where $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$, has a derivative for each real x given by $p'(x) = a_1 + 2a_2x + \dots + na_nx^{n-1}$.

Proof. $p'(x) = 0 + a_1Dx + a_2Dx^2 + \dots + a_nDx^n$ (Theorem 4-2a).

$= a_1 + 2a_2x + \dots + na_nx^{n-1}$. (Theorem 4-2c).

Corollary 3. If p is a polynomial and if f' exists then $D_xp(f(x)) = p'(f(x))\ f'(x)$.

Proof. $p(f(x)) = a_0 + a_1(f(x)) + a_2(f(x))^2 + \dots + a_n(f(x))^n$, and

$$D_xp(f(x)) = a_1Df(x) + a_2D(f(x))^2 + \dots + a_nD(f(x))^n$$

$$= a_1f'(x) + 2a_2f(x)f'(x) + \dots + n\,a_n(f(x))^{n-1}f'(x)$$

$$= \left(a_1 + 2a_2f(x) + \dots + n\,a_n(f(x))^{n-1}\right)\ f'(x).$$

But

$D_yp(y) = p'(y) = a_1 + 2a_2y + \dots + n\,a_n(y)^{n-1}$ where $y = f(x)$ so

$D_xp(f(x)) = p'(f(x))\ f'(x)$.

2. Evaluate:

(a) $D(x^5 - x^8 + x^{11}) = 5x^4 - 8x^7 + 11x^{10}$.

(b) $D(5 + x)^6 = 6(5 + x)^5$.

(c) $D(3 - 2x^3)^5 = -30x^2(3 - 2x^3)^4$.

(d) $D(x - 3x^2 + 5x^3)^4 = 4(1 - 6x + 15x^2)(x - 3x^2 + 5x^3)^3$.

(e) $D\left(x(1 - x^2)^3\right) = (1 - x^2)^2(1 - 7x^2)$.

(f) $D\left((3 - 5x)^2(1 - x^2)^3\right) = (3 - 5x)(1 - x^2)^2(40x^2 - 18x - 10)$.

(g) $D(1 - \frac{1}{x})^3 = \frac{3}{x^2}(1 - \frac{1}{x})^2$.

(h) $D(3x^{1/2} - 4x^{3/2})^6 = 6(3x^{1/2} - 4x^{3/2})^5(\frac{3}{2}x^{-1/2} - 6x^{1/2})$.

(i) $D(1 - \sqrt{x})^{10} = -\frac{5}{\sqrt{x}}(1 - \sqrt{x})^9$.

(j) $D\left((3 - 5x + x^2)^3(1 + x^2)^{10}\right)$

$= (1 + x^2)^9(3 - 5x + x^2)^2(-15 + 66x - 105x^2 + 26x^3)$.

3. Consider the curves $y = ax^3 + 1$ and $y = bx^2$.

(a) Find two numbers a and b such that the curves have the same slope at $x = 1$ and that the sum of the slopes at $x = 2$ is 36 .

For $y = ax^3 + 1$, $D_x y = 3ax^2$ and

for $y = bx^2$, $D_x y = 2bx$.

$3a = 2b$ and $3a(4) + 2b(2) = 36$.

Thus, $a = 2$, $b = 3$.

(b) Find values of a and b such that the curves have the same slope at a point of intersection.

At points of intersection

$$ax^3 + 1 = bx^2 .$$

For equal slopes at these points,

$$3ax^2 = 2bx .$$

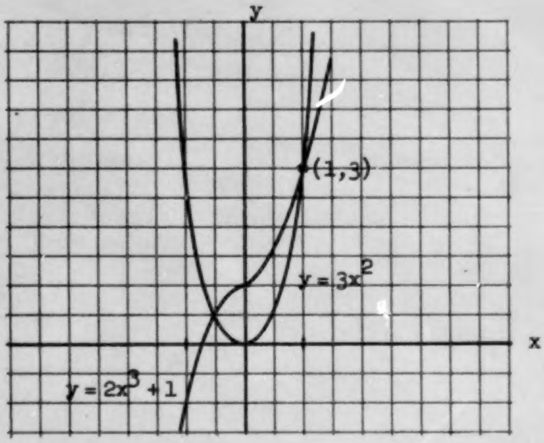
Since we have two conditions and three unknowns (a , b , and x) , we can assign x arbitrarily. Then,

$$\frac{ax^3 + 1}{3ax^2} = \frac{x}{2}$$

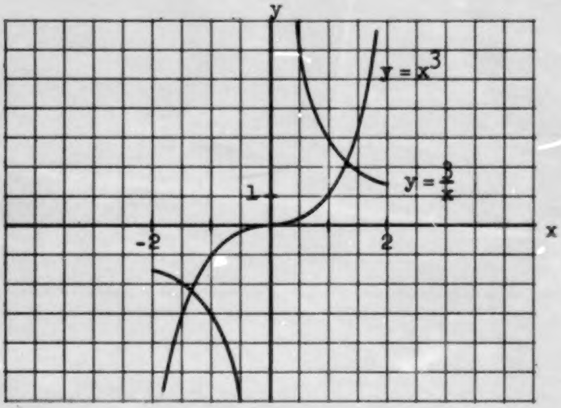
and $a = \frac{2}{x^3}$, $b = \frac{3}{x^2}$.

For $x = 1$, $a = 2$, $b = 3$.

(c) Sketch the curves in part (b) for some allowable values of a and b.



4. (a) Sketch the curves $y = x^3$, $y = \frac{3}{x}$ for $|x| \leq 2$.



(b) For what values of x is $g'(x) = 0$, where $g(x) = x^3 + \frac{3}{x}$ for $|x| \leq 2$?

$$g'(x) = 3x^2 - \frac{3}{x^2}$$

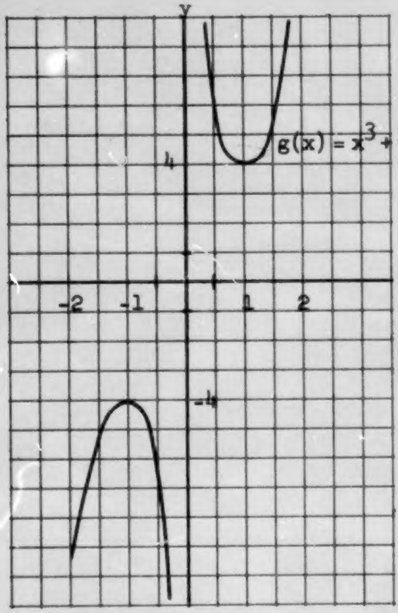
$g'(x) = 0$ if $x = \pm 1$.

(c) **Find the discontinuities of g and g'.**

g is discontinuous at $x = 0$.

g' is discontinuous at $x = 0$.

(d) **Using the results of a, b, c, sketch the graph of g for $-2 < x < 2$.**



5. Let $u = f(x)$, $v = g(x)$, and $w = h(x)$.

(a) Prove that if the functions f, g, and h are differentiable at x, then

$$D(uvw) = (uv)Dh(x) + (uw)Dg(x) + (vw)Df(x) .$$

$$\begin{aligned} D_x(uv)(w) &= (uv)D_xh(x) + h(x)D_x(uv) \\ &= (uv)D_xh(x) + h(x)[u\, D_xg(x) + v\, D_xf(x)] \\ &= (uv)D_xh(x) + (uw)D_xg(x) + (vw)D_xf(x) . \end{aligned}$$

(b) Can you suggest a way to generalize your result to obtain a formula for the derivative of a product of n functions? Test your conjecture with the case $n = 4$.

Let $F = f_1 \cdot f_2 \cdot f_3 \cdot \ldots \cdot f_n$ then

$$DF = f_1' \cdot f_2 \cdot f_3 \cdot \ldots \cdot f_n + f_1 \cdot f_2' \cdot f_3 \cdot \ldots \cdot f_n + \ldots + f_1 \cdot f_2 \cdot f_3 \cdot \ldots \cdot f_n' .$$

For $n = 4$, let $z = k(x)$. Then

$$D_x(uvw)(z) = (uvw)D_xk(x) + z\,D_x(uvw)$$

$$= (uvw)D_xk(x) + (uvz)D_xh(x) + (uwz)D_xg(x) + (vwz)D_xf(x) .$$

(c) Use the above result to evalute:

(i) $D[(5x - 2)(3 - 2x)(x^2 + 1)]$

$$= 2x(5x - 2)(3 - 2x) - 2(5x - 2)(x^2 + 1) + 5(3 - 2x)(x^2 + 1)$$

(ii) $D[(2x^3 - 3x^2 + 1)(\sqrt{x} + 1)^2]$

$$= \frac{1}{\sqrt{x}}(2x^3 - 3x^2 + 1)(\sqrt{x} + 1) + (\sqrt{x} + 1)^2(6x^2 - 6x)$$

(iii) $D[(3x - 2)(1 - x^2)(1 + x)(1 + x^2)]$

$$= 2x(3x - 2)(1 - x^2)(1 + x) + (3x - 2)(1 - x^2)(1 + x^2)$$

$$-2x(3x - 2)(1 + x)(1 + x^2) + 3(1 - x^2)(1 + x)(1 + x^2)$$

or $(1 + x)(1 + x^2)(3 + 4x - 9x^2) + (3x - 2)(1 - x^2)(1 + 2x + 3x^2)$

Solutions Exercises 4-2d

1. Evaluate:

(a) $D\left(\frac{x}{x - 1}\right) = \frac{-1}{(x - 1)^2}$

(b) $D\left(\frac{x^2}{1 + x^2}\right) = \frac{2x}{(1 + x^2)^2}$

(c) $D\left(1 - \frac{1}{x}\right)^{-1} = D\left(\frac{x}{x - 1}\right) = \frac{-1}{(x - 1)^2}$

(d) $D\left(\frac{3 + 2x^2}{2 - x^2}\right) = \frac{14x}{(2 - x^2)^2}$

(e) $D\left(\frac{1}{x} + \frac{1}{1 - x}\right) = \frac{2x - 1}{x^2(1 - x)^2}$

(f) $D\left(\frac{\sqrt{x}}{1 + x^2}\right) = \frac{1 - 3x^2}{2\sqrt{x}(1 + x^2)^2}$

(g) $D\left(\frac{1}{1 + \sqrt{x}}\right) = -\frac{1}{2\sqrt{x}(1 + \sqrt{x})^2}$

(h) $D\left(\frac{x^2 - 1}{x^2 + 1}\right)^{-1} = D\left(\frac{x^2 + 1}{x^2 - 1}\right) = \frac{-4x}{(x^2 + 1)^2}$

(i) $D\left(\frac{1}{[\sqrt{x}]}\right) = 0$, $x \neq n^2$, n an integer.

(j) $D\left(\frac{1}{\sqrt{[x]}}\right) = 0$, $x \neq n$, n an integer.

2. Prove Corollary 1 to Theorem 4-2d.

If f and g have derivatives at x and $g'(x) \neq 0$, then for the quotient $G(x) = \frac{f(x)}{g(x)}$ we have

$$G'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} .$$

__Proof__.

$$\begin{aligned} D\left(\frac{f(x)}{g(x)}\right) &= D\left(f(x) \cdot \frac{1}{g(x)}\right) \\ &= f(x)D\frac{1}{g(x)} + \frac{1}{g(x)} Df(x) \\ &= f(x)\left(\frac{-g'(x)}{(g(x))^2}\right) + \frac{1}{g(x)} \cdot f'(x) \\ &= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} \end{aligned}$$

3. **Evaluate:**

(a) $D\left(\frac{x^2 + 5}{x^2 - 5}\right) = \frac{-20x}{(x^2 - 5)^2}$

(b) $D\left(\frac{x + 1}{x} - \frac{x^2 + 1}{x^2}\right) = \frac{x^3 + x + 2}{x^3}$

(c) $D\left(\frac{x - 1}{x^2 + x + 1}\right) = \frac{-x^2 + 2x + 2}{(x^2 + x + 1)^2}$

(d) $D\left((1 + \frac{1}{x})(x + 1)\right) = x + 2 - \frac{1}{x^2}$

(e) $D\left(\frac{1 - x + x^2}{1 + x + x^2}\right) = \frac{2(x^2 - 1)}{(1 + x + x^2)^2}$

(f) $D\left(\frac{1 - x}{x^2(1 + x^2)}\right) = \frac{3x^3 - 4x^2 + x - 2}{x^3(1 + x^2)^2}$

(g) $D\left(\frac{1 + x^{-1}}{1 - x^{-1}}\right) = \frac{-2}{(x - 1)^2}$

(h) $D\left(\frac{ax^2 + bx + c}{dx^2 + ex + f}\right) = \frac{(ae - bd)x^2 + 2(af - dc)x + bf - ce}{(dx^2 + ex + f)^2}$

(i) $D\left(\frac{x^4 + x^2 + 1}{x^2 + x + 1}\right) = D(x^2 - x + 1) = 2x - 1$

(j) $D\left(\frac{x^5 + x + 1}{x^2 + x + 1}\right) = D(x^3 - x^2 + 1) = 3x^2 - 2x$

4. **Determine both**

$$D\left(\frac{ax + b}{cx + d}\right) \quad \text{and} \quad D\left(\frac{bc - ad}{c(cx + d)}\right).$$

Explain why both are the same.

$$D\left(\frac{ax + b}{cx + d}\right) = \frac{ad - bc}{(cx + d)^2} = D\left(\frac{bc - ad}{c(cx + d)}\right).$$

Since $\frac{ax + b}{cx + d} = \frac{a}{c} + \frac{bc - ad}{c(cx + d)}$ the derivatives are equal.

5. **Find the derivative of each of the following functions in as many ways as you can and describe its domain. (Do not overlook definition of derivative.)**

(a) $f : x \longrightarrow \frac{|x|}{x}$, $f'(x) = 0$, $x \neq 0$.

(b) $f : x \longrightarrow \frac{x}{[x]}$, $f'(x) = \frac{1}{[x]}$, x not an integer.

(c) $f : x \longrightarrow \frac{x - [x]}{|x|}$, $f'(x) = \frac{[x]}{x^2}$, x not an integer.

(d) $f : x \longrightarrow \frac{(x - [x])^2}{|x|}$, $f'(x) = \frac{|x|(2x - x^2 - [x]^2)}{x^3}$, x not an integer.

6. Prove Corollary 2 to Theorem 4-2d.

If R is a rational function, if $R\big(f(x)\big)$ is defined, and if $f'(x)$ exists, then

$$D_x R\big(f(x)\big) = R'\big(f(x)\big) \cdot f'(x) .$$

Let $R(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, so that

$$R\big(f(x)\big) = \frac{p(f(x))}{q(f(x))}, \quad q\big(f(x)\big) \neq 0 .$$

$$D_x p\big(f(x)\big) = p'\big(f(x)\big) \cdot f'(x)$$

$$D_x q\big(f(x)\big) = q'\big(f(x)\big) \cdot f'(x) .$$

$$\begin{aligned} D_x R\big(f(x)\big) &= \frac{q\big(f(x)\big)\ D_x p\big(f(x)\big) - p\big(f(x)\big)\ D_x q\big(f(x)\big)}{[q(f(x))]^2} \\ &= \frac{q(f(x))\ p'(f(x))\ f'(x) - p(f(x))\ q'(f(x))\ f'(x)}{[q(f(x))]^2} \\ &= f'(x)\ \frac{q(f(x))\ p'(f(x)) - p(f(x))\ q'(f(x))}{[q(f(x))]^2} \\ &= R'\big(f(x)\big) \cdot f'(x) . \end{aligned}$$

7. Evaluate

(a) $D_x\left(\frac{(x^2 - x^3)^2 - (x^2 - x^3)^4}{(x^2 - x^3)^2 + 1}\right)^7$ Set $u = x^2 - x^3$, $v = D_x(x^2 - x^3)$

$= 2$[illegible]$- 3x^2$

$$D_x\left(\frac{u^2 - u^4}{u^2 + 1}\right)^7 = v\left[\frac{14u^3(1 - u^2)^6(1 - 2u^2 - u^4)}{(u^2 + 1)^8}\right]$$

(b) $D_x\left(\frac{5(1 - \sqrt{x}) + 3}{(1 - \sqrt{x})^2 - 3}\right)^2$. Set $u = 1 - \sqrt{x}$, $v = D_x(1 - \sqrt{x})$

$= 1 - \frac{1}{2\sqrt{x}}$

$$D_x\left(\frac{5u + 3}{u^2 - 3}\right)^2 = -2v\left[\frac{(5u + 3)(5u^2 + 6u + 15)}{(u^2 - 3)^2}\right]$$

(c) $D_x\left(\frac{(\sqrt{x} - \frac{1}{x})^4 - 1}{(\sqrt{x} - \frac{1}{x})^2 + 1}\right)^3$ Set $u = \sqrt{x} - \frac{1}{x}$, $v = D_x(\sqrt{x} - \frac{1}{x})$

$= \frac{1}{2\sqrt{x}} + \frac{1}{x^2}$

$$D_x\left(\frac{u^4 - 1}{u^2 + 1}\right)^3 = D_x(u^2 - 1)^3 = 6vu(u^2 - 1)^2$$

8. Consider the quotient $\phi(x) = \frac{f(x)}{g(x)}$ where f and g have derivatives at [illegible] and where $g(x) \neq 0$. Obtain the formula for the derivative of a quotient by applying the product rule (Theorem 4-2b) to the expression $\phi(x)g(x) = f(x)$.

$\phi(x)g'(x) + g(x)\phi'(x) = f'(x)$, by Theorem 4-2b

$\phi'(x) = \frac{f'(x) - \phi(x)g'(x)}{g(x)}$, solving for $\phi'(x)$

$= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$, by given condition.

Why does this not constitute a proof of the quotient rule (Corollary 1 to Theorem 4-2d)?

This is not a proof because it assumes the existence of $\phi'(x)$.

TC4-3. Inverse Functions, Fractional Powers.

Section A2-3 includes a definition and brief discussion of inverse functions. More serious consideration of the idea is given in Section TC A2-3. If f and g are inverses of each other, then f is the inverse of g and g is the inverse of f. The definition (Definition A2-3b) requires that the domain of the inverse of a function f be the range of f: for example, while $(\sqrt{x})^2 = x$ for all x in the domain of $f : x \longrightarrow \sqrt{x}$, the function $g : x \longrightarrow x^2$ is not the inverse of f (since the domain of g is not the range of f).

For a discussion of fractional powers see Sections TC A2-1 (Example A2-1b) and TC A2-4.

Solutions Exercises 4-3a

1. Show that $f : x \longrightarrow \frac{1 - x}{1 + x}$, where $x > -1$, has an inverse. Find an equation that defines the inverse g and find the derivative of g.

If $y = f(x) = \frac{1 - x}{1 + x}$ and $x > -1$, then $x = \frac{1 - y}{y + 1}$.

The inverse function g is defined by $y = g(x) = \frac{1 - x}{1 + x}$, $x > -1$.

Then

$$f(y) = \frac{1 - \left(\frac{1 - x}{1 + x}\right)}{1 + \left(\frac{1 - x}{1 + x}\right)} = \frac{2x}{2} = x \ , \ x > -1 \ .$$

$$D_x g(x) = D_x\left(\frac{1 - x}{1 + x}\right) = \frac{-2}{(1 + x)^2}$$

2. Verify that the inverse of $f : x \longrightarrow x \cdot |x|$ exists and then find the derivative of the inverse.

$$f(x) = \begin{cases} x^2 & , x \geq 0 \\ -x^2 & , x < 0 \end{cases}$$

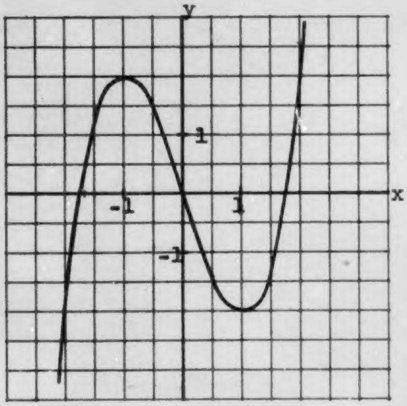
f is strongly monotone; hence it has an inverse g, and

$$g(x) = \begin{cases} \sqrt{x} & , x \geq 0 \ , \\ -\sqrt{-x} & , x < 0 \ , \end{cases} \quad \text{and} \quad \begin{array}{l} D_x\sqrt{x} = \dfrac{1}{2\sqrt{x}} \ , \ x > 0 \ , \\ D_x -\sqrt{-x} = \dfrac{1}{2\sqrt{-x}} \ , \ x < 0 \ . \end{array}$$

Hence if g is the inverse of $f : x \longrightarrow x|x|$,

$$D_x g(x) = \frac{1}{2\sqrt{|x|}} , \text{ for all } x \neq 0 .$$

3. Sketch the graph of $f : x \longrightarrow x^3 - 3x$ and tell why f does not have an inverse. Indicate how you can divide the domain of f into three parts and define three new functions each of which agrees with f on its domain and has an inverse. Justify your result. (See Exercises 2-3, No. 6)



If f has an inverse, no line parallel to the x-axis may meet the graph of f in more than one point.

$f : x \longrightarrow x^3 - 3x$

$f' : x \longrightarrow 3x^2 - 3 = 3(x - 1)(x + 1)$

Each of the following functions has an inverse.

$f : x \longrightarrow x^3 - 3x , \quad x < -1$

$f : x \longrightarrow x^3 - 3x , \quad -1 \leq x \leq 1$

$f : x \longrightarrow x^3 - 3x , \quad x > 1 .$

4. Consider the function defined by

$$f(x) = \begin{cases} 2x & , \text{ for } x > 0 \text{ and irrational,} \\ x^2 + 1 & , \text{ for } x \geq 0 \text{ and rational.} \end{cases}$$

Show that f has a derivative at $x = 1$. Prove that the mapping f is one-to-one; i.e., that f has an inverse. Prove that the inverse of f is not differentiable at any point other than the point $y = 2$ of its domain.

(Note: The statement of the problem in the text is incomplete.)

(i) Show that f has a derivative at $x = 1$.

$$\frac{f(x) - 2}{x - 1} - 2 = \begin{cases} 0 & , \text{ for } x > 0 \text{ and irrational,} \\ x - 1 & , \text{ for } x > 0 \text{ and rational .} \end{cases}$$

Hence, for all x , $\left|\frac{f(x) - 2}{x - 1} - 2\right| \leq |x - 1|$ and therefore, given $\epsilon > 0$, if $0 < |x - 1| < \epsilon$, then $\left|\frac{f(x) - 2}{x - 1} - 2\right| < \epsilon$. Hence $f'(1) = 2$.

(ii) Prove that f is one-to-one.

Since $2x_1 = x_2^2 + 1$, x_1 irrational, x_2 rational, is impossible because the left-hand side is irrational while the right-hand side is rational, it follows that $f(x_1) = f(x_2)$ implies that x_1 and x_2 are both rational or both irrational. Hence, either $2x_1 = 2x_2$, which implies $x_1 = x_2$, or $x_1^2 + 1 = x_2^2 + 1$ which, for $x \geq 0$, implies $x_1 = x_2$. Hence, $f(x_1) = f(x_2)$ implies $x_1 = x_2$ and f is one-to-one.

(iii) Prove that the inverse of f is differentiable only at the point $y = 2$ of its domain.

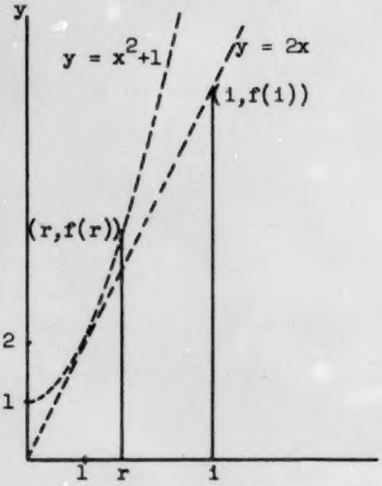
The inverse of f is

$$g : y \longrightarrow \begin{cases} \frac{1}{2}y & \text{, for } y > 0 \text{ and irrational,} \\ \sqrt{y-1} & \text{, for } y \in A = \{y : y \geq 1 \text{ and } y - 1 \text{ is the} \\ & \qquad \text{square of a rational number}\} . \end{cases}$$

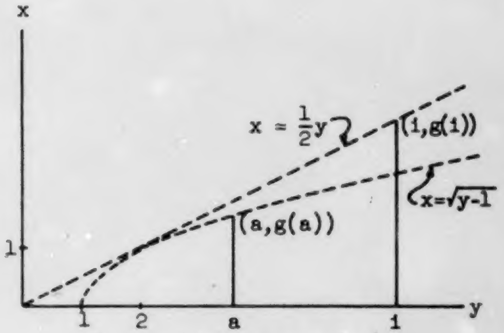
Then

$$\frac{g(y) - 1}{y - 2} - \frac{1}{2} = \begin{cases} 0 \text{ , for } y \text{ irrational,} \\ \dfrac{2 - y}{2(\sqrt{y-1} + 1)^2} \text{ , } y \in A \text{ and } y \neq 2 . \end{cases}$$

Hence, for all y in the domain of g other than 2, $\left|\frac{g(y) - 1}{y - 2} - \frac{1}{2}\right| \leq |y - 2|$ and therefore, given $\epsilon > 0$, if $0 < |y - 2| < \epsilon$, then $\left|\frac{g(y) - 1}{y - 2} - \frac{1}{2}\right| < \epsilon$. Thus, $g'(2) = \frac{1}{2}$.



(r rational, i irrational)



(a ∈ A, i irrational)

Now let $h : y \longrightarrow \frac{1}{2} y - \sqrt{y - 1}$ for all $y \geq 1$. Since $h(y)$ is increasing for $y > 2$ and decreasing for $1 < y < 2$ (see graph), it follows that if $|y_1 - 2| > \delta$ and $|y_2 - 2| > \delta$, then $|\frac{1}{2} y_1 - \sqrt{y_2 - 1}| > k = \min\{h(2 - \delta), h(2 + \delta)\}$. Furthermore, given any y_1 irrational (or $y_1 \in A$ and $y_1 \neq 2$), we can find y_2 in A (or y_2 irrational) as close as we wish to y_1. But so long as $|y_1 - y_2| < \frac{|y_1 - 2|}{2} = \delta$, $|g(y_1) - g(y_2)| > k$. Hence g is not continuous at y_1 and therefore not differentiable at y_1.

Solutions Exercises 4-3b

1. Evaluate the following and express your answers using positive exponents only.

(a) $Dx^{2/3} = \frac{2}{3x^{1/3}}$

(b) $Dx^{3/5} = \frac{3}{5x^{2/5}}$

(c) $Dx^{-2/3} = -\frac{2}{3x^{5/3}}$

(d) $Dx^{-3/5} = -\frac{3}{5x^{8/5}}$

2. Find $f'(2)$ if:

(a) $f(x) = (2x)^{1/3}$

$f'(x) = \frac{\sqrt[3]{2}}{3} x^{-2/3}$

$f'(2) = \frac{1}{3\sqrt[3]{2}}$

(b) $f(x) = x^{-1/4}$, $x > 0$

$f'(x) = -\frac{1}{4} x^{-5/4}$

$f'(2) = -\frac{1}{8\sqrt[4]{2}}$

(c) $f(x) = x^{-4/5}$

$f'(x) = -\frac{4}{5} x^{-9/5}$

$f'(2) = -\frac{2}{5\sqrt[5]{16}}$

3. **Evaluate:**

(a) $D(x^{1/3} - 3x^{-1/3}) = \frac{1}{3}x^{-2/3} + x^{-4/3}$.

(b) $$D\left(\frac{1 + x^{4/3}}{1 - x^{4/3}}\right) = \frac{\frac{8}{3}x^{1/3}}{(1 - x^{4/3})^2} .$$

(c) $$d\left(\sqrt{x^3} - \sqrt{x} + \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x^3}}\right) = \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} + \frac{3}{2}x^{-5/2} .$$

4. Let $f(x) = x^n$ for n an integer. For what points a in the domain of f do the hypotheses of Theorem 4-3 hold?

If $n = 0$, $f(x) = 1$ and $f'(x) = 0$. Theorem does not apply.

If $n = 1$, $f(x) = x$ and $f'(x) = 1$. Theorem holds for all a .

If $n > 0$ and even, f is increasing for $x \geq 0$, decreasing for $x < 0$, and $f'(0) = 0$. If $n > 0$ and odd, f is increasing for all x, but $f'(0) = 0$. For $n < 0$, $x = 0$ is not in the domain of f and f is decreasing for $x > 0$ and either increasing or decreasing for $x < 0$. The Theorem holds for all a except zero.

5. Consider the function f : [illegible] $\longrightarrow \sqrt[n]{x}$. We have $f(x)^n = x$. Applying Theorem 4-2c, obtain

$$D\left(\sqrt[n]{x}\right) = \frac{1}{n}x^{\frac{1}{n} - 1} .$$

Why is this not a proof of the corollary to Theorem 4-3?

$$D_x\, f(x)^n = n\, f(x)^{n-1} f'(x) = D_x(x) = 1$$

Thus,

$$f'(x) = \frac{1}{n\, f(x)^{n-1}} = \frac{1}{n(x^{1/n})^{n-1}} = \frac{1}{n}x^{\frac{1}{n} - 1}$$

This does not constitute a proof of the Corollary since it was assumed that $f'(x)$ exists in applying Theorem 4-2c.

6. Let $r = \frac{p}{q}$ under the hypothesis of the corollary to Theorem 4-3 and let q be odd. Prove the corollary for $x < 0$. Why is the case $x = 0$ not included?

(Corollary: For every rational number r, $Dx^r = rx^{r-1}$.)

Let $h(x) = x^{1/q}$. For $x < 0$, $x^{1/q} = y < 0$. $f(y) = y^q$ is increasing, and $f'(y) = qy^{q-1} \neq 0$.

Thus, since h is the inverse of f, $h'(x) = \frac{1}{f'(y)} = \frac{1}{q}x^{\frac{1}{q} - 1}$, by Theorem 4-3.

Now, Theorem 4-2c applies, and

$$\begin{aligned} D(x^{p/q}) &= D(x^{1/q})^p \\ &= p(x^{1/q})^{p-1} \frac{1}{q} x^{\frac{1}{q} - 1} \\ &= \frac{p}{q} x^{\frac{p}{q} - \frac{1}{q} + \frac{1}{q} - 1} = \frac{p}{q} x^{\frac{p}{q} - 1} \end{aligned}$$

At $x = 0$, $f'(0) = 0$, so Theorem 4-3 does not apply, and $\lim_{h \to 0} \frac{h^{p/q}}{h}$ does not exist if $p < q$.

7. Under the conditions of the preceding exercise, show that for $g : x \longrightarrow x^{p/q}$, where $p > q$, the derivative of g at zero exists and $g'(0) = 0$.

$$\frac{g(0 + h) - g(0)}{h} = \frac{h^{p/q}}{h} = h^{\frac{p}{q} - 1} .$$

Since $p > q$, $\frac{p}{q} > 1$, so $\frac{p}{q} - 1 > 0$.

Thus, if $\epsilon > 0$ is given, let $\delta = \epsilon^{\frac{q}{p-q}}$

Then if $0 < |h| < \delta$,

$$\left|\frac{g(0 + h) - g(0)}{h}\right| = |h|^{\frac{p}{q} - 1} < \epsilon .$$

So, $$\lim_{h \to 0} \frac{g(0 + h) - g(0)}{h} = 0 = g'(0) .$$

TC4-4. Circular Functions.

In the discussion of circular functions, a student may wonder why we appeal to a geometric discussion to evaluate $\lim\limits_{h \to 0} \frac{\sin h}{h}$. In his study of trigonometry he employed a diagram to define circular functions and to obtain the basic relationships of Section A2-5. He might expect that we are now in a position to evaluate this limit by formal computation from the relationships of Section A2-5 with no further appeal to intuitive geometric discussion. The reason this cannot be done is that the trigonometric identities of Section A2-5 have the same form, no matter what unit is used for angle measure. The crucial inequality

$$\cos x < \frac{\sin x}{x} < 1$$

is the first place where angle measure appears both as an argument of a trigonometric function and independently. Thus the inequality depends on the unit of measure; it holds specifically for radian measure and does not hold for any other measure such as degree measure.

You may wish to refer to Section A2-5 and/or Section TC A2-5.

Solutions Exercises 4-4

1. Show that $D \cos x = -\sin x$.

$$\begin{aligned} D \cos x &= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} \\ &= \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \cos x \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \to 0} \frac{\sin h}{h} \\ &= -\sin x . \quad \left(\text{See text for proof of } \lim_{h \to 0} \frac{\cos h - 1}{h} = 0 .\right) \end{aligned}$$

2. Evaluate $\lim\limits_{h \to 0} \frac{\tan h}{h}$. (Hint: Express $\tan h$ in terms of $\sin h$ and $\cos h$.)

$$\lim_{h \to 0} \frac{\tan h}{h} = \lim_{h \to 0} \frac{\sin h}{h} \cdot \frac{1}{\cos h} = 1 .$$

3. **From the definition of the derivative as a limit and the result of Number 2 derive the formula**

$$D \tan x = \sec^2 x \, .$$

[Hint: $\tan(x + h) = \dfrac{\tan x + \tan h}{1 - \tan x \tan h}$.]

$$D \tan x = \lim_{h \to 0} \frac{\tan(x + h) - \tan x}{h} = \lim_{h \to 0} \frac{\tan h + \tan^2 x \tan h}{h(1 - \tan x \tan h)}$$

$$= \lim_{h \to 0} \frac{\tan h}{h} \cdot \lim_{h \to 0} \frac{1 + \tan^2 x}{1 - \tan x \tan h}$$

$$= \sec^2 x$$

Compare this result with the result obtained by using the method of differentiating the quotient $\dfrac{\sin x}{\cos x}$.

$$D \tan x = D \frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} \, .$$

4. In the simplest way you can, evaluate the following and express your answers in several different equivalent forms.

(a) $D \cot x = D \dfrac{1}{\tan x} = \dfrac{-D \tan x}{\tan^2 x} = - \dfrac{\sec^2 x}{\tan^2 x} = \dfrac{-1}{\sin^2 x}$

$$= - \csc^2 x = -1 - \cot^2 x \, .$$

(b) $D \sec x = D \dfrac{1}{\cos x} = \dfrac{-D \cos x}{\cos^2 x} = \dfrac{\sin x}{\cos^2 x} = \dfrac{\tan x}{\cos x}$

$$= \tan x \sec x = \sin x \sec^2 x$$

(c) $D \csc x = D \dfrac{1}{\sin x} = \dfrac{-\cos x}{\sin^2 x} = \dfrac{-\cot x}{\sin x} = - \cot x \csc x$

$$= - \cos x \csc^2 x$$

(d) $D \sin^2 x = 2 \sin x \, D \sin x = 2 \sin x \cos x = \sin 2x$

(e) $D \cos^2 x = 2 \cos x \cdot (-\sin x) = -2 \sin x \cos x = -\sin 2x$,

or $D \cos^2 x = D(1 - \sin^2 x) = -\sin 2x$

(f) $$D(4 \cos^3 x - 3 \cos x) = -12 \cos^2 x \sin x + 3 \sin x$$
$$= 3 \sin x(1 - 4 \cos^2 x)$$
$$= 3 \sin x(4 \sin^2 x - 3)$$

Check: $D(\cos 3x) = -3 \sin 3x$

(g) $$D(3 \sin x - 4 \sin^3 x) = 3 \cos x - 12 \sin^2 x \cos x$$
$$= 3 \cos x(4 \cos^2 x - 3)$$

Check: $D(\sin 3x) = 3 \cos 3x$

5. Evaluate the following limits:

(a) $$\lim_{h \to 0} \frac{\sin 2h}{h} = \lim_{h \to 0} \frac{2 \sin h \cos h}{h}$$
$$= \lim_{h \to 0} \frac{\sin h}{h} \cdot \lim_{h \to 0} (2 \cos h)$$
$$= 1 \cdot 2 = 2$$

(b) $$\lim_{h \to 0} \frac{1 - \cos h}{h^2} = \lim_{h \to 0} \frac{1 - \cos^2 h}{h^2(1 + \cos h)}$$
$$= \lim_{h \to 0} \frac{\sin^2 h}{h^2} \cdot \frac{1}{1 + \cos h}$$
$$= 1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2}$$

6. (a) Given that $\lim_{x \to 0} \sin x = \sin 0 = 0$.

Prove that $\lim_{x \to 0} \frac{\sin x}{\cos x + 1} = 0$. (Hint: Show that $\cos x$ is continuous at $x = 0$).

Since $\cos x = 1 - 2 \sin^2 \frac{x}{2}$,

$$\lim_{x \to 0} \cos x = \lim_{x \to 0} (1 - 2 \sin^2 \frac{x}{2})$$
$$= 1 - 2 \lim_{x \to 0} \sin^2 \frac{1}{x}$$
$$= 1 .$$

Thus, cos x is continuous at $x = 0$.

$$\lim_{x\to 0} \frac{\sin x}{1 + \cos x} = \frac{\lim_{x\to 0} \sin x}{\lim_{x\to 0} (1 + \cos x)} = 0$$

(b) From the preceding result prove that sin x and cos x are continuous for all values of x.
Make explicit just what is being assumed in the proofs of (a) and (b).

To prove that sin x and cos x are continuous for all real values of x, we use the formula,

$$\sin(x + h) = \sin x \cos h + \cos x \sin h .$$

Thus,
$$\begin{aligned}\lim_{h\to 0} \sin(x + h) &= \lim_{h\to 0}(\sin x \cos h) + \lim_{h\to 0}(\cos x \sin h)\\ &= \sin x \cdot 1 + \cos x \cdot 0\\ &= \sin x\end{aligned}$$

This proves that sin x is continuous for all real x.

Since $\cos x = \sin(x + \frac{\pi}{2})$, cos x is also continuous for all x.

Observe that in addition to $\lim_{x\to 0} \sin x = 0$, we needed no statement involving limits other than limit theorems. Thus we have assumed only this limit and conventional identities for circular functions.

7. Given that $Df(x) = G(x)$, show that $Df(ax + b) = aG(ax + b)$, provided f is differentiable at $ax + b$.

Comment: At first glance this problem may appear to be a simple application of the chain rule but at this point the chain rule has not been established and the proof of the problem will require the definition of the derivative.

(1) $$Df(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h} = G(x)$$

(2) $$\begin{aligned}Df(ax + b) &= \lim_{k\to 0} \frac{f(a(x + k) + b) - f(ax + b)}{k}\\ &= a \cdot \lim_{\ell\to 0} \frac{f(ax + b + \ell) - f(ax + b)}{\ell}\end{aligned}$$

where $\ell = ak$ and $a \neq 0$.

By (1) the limit in (2) is $aG(ax + b)$.

8. **Evaluate the following:**

(a) $D \cos^2 x \sin 2x = 2\,D \cos^3 x \sin x$

$$= 2\cos^2 x(4\cos^2 x - 3)\ .$$

Alternatively, using Number 7,

$$D \cos^2 x \sin 2x = \cos^2 x(2\cos 2x) - \sin 2x(2\cos x \sin x)\ .$$

(b) $D \sin^2 (ax + b) = 2 \sin (ax + b)\, D \sin (ax + b)$.

By Number 7, if $D \sin x = \cos x$ then $D \sin (ax + b) = a \cos (ax + b)$

$$D \sin^2 (ax + b) = 2a \sin (ax + b) \cos (ax + b)\ .$$

(c) $D(\sin 7x)(\cos 2x) = \sin 7x\, D \cos 2x + \cos 2x\, D \sin 7x$.

By Number 7, $D \cos 2x = -\,2 \sin 2x$

and, $D \sin 7x = 7 \cos 7x$.

$D(\sin 7x)(\cos 2x) = -\,2 \sin 2x \sin 7x + 7 \cos 2x \cos 7x$.

9. Let $g(x) = |\cos x|$. Discuss the domain of the derivative for x in the interval $0 \le x \le \pi$.

$$g(x) = \begin{cases} \cos x\ , & 0 \le x \le \frac{\pi}{2}\ , \\ -\cos x\ , & \frac{\pi}{2} < x \le \pi\ . \end{cases}$$

$$g'(x) = \begin{cases} -\sin x\ , & 0 \le x < \frac{\pi}{2}\ , \\ \sin x\ , & \frac{\pi}{2} < x \le \pi\ . \end{cases}$$

or

$$g'(x) = -\sin x \,\frac{|\cos x|}{\cos x}\ .$$

$g'(x)$ is not defined at $x = \frac{\pi}{2}$.

10. Find a point on the graph of $y = \sin x$ at which the slope of the curve is equal to the slope of the line $x + 2y + 2 = 0$. Is there only one such point? Justify your answer.

The slope of the graph of f at any point $(x, f(x))$ is $f'(x) = \cos x$. The line $x + 2y + 2 = 0$ has slope $-\frac{1}{2}$. If $x = \pm\frac{2}{3}\pi + 2n\pi$, n an integer, then $\cos x = -\frac{1}{2}$. Also, by periodicity, if there is one such point there are an infinite number of such points.

11. Evaluate the following:

(a) $$D\left(\frac{1 - \sin x}{1 + \cos^2 x}\right) = \frac{(1 + \cos^2 x)(-\cos x) + (1 - \sin x)(2 \cos x \sin x)}{(1 + \cos^2 x)^2}$$

$$= \frac{\sin 2x - 3 \cos x + \cos^3 x}{(1 + \cos^2 x)^2}$$

(b) $$D\left(\frac{1 - \tan^2 x}{2 \tan x}\right) = \frac{2 \tan x(-2 \tan x \sec^2 x) - (1 - \tan^2 x)(2 \sec^2 x)}{4 \tan^2 x}$$

$$= \frac{-1}{2 \cos^2 x \sin^2 x}$$

(c) $$D\left(\frac{\sin^4 x}{\cos^2 x}\right) = \frac{\cos^2 x \; 4 \sin^3 x \cos x + \sin^4 x \; 2 \cos x \sin x}{\cos^4 x}$$

$$= \frac{2 \sin^3 x \; (2 \cos^2 x + \sin^2 x)}{\cos^3 x}$$

(d) $$D\left(\frac{3}{\cos^2 x}\right) = \frac{-3(2 \cos x(-\sin x))}{\cos^4 x} = \frac{6 \sin x}{\cos^3 x}$$

(e) $$D\left(\frac{1}{1 + \tan x}\right) = \frac{-\sec^2 x}{(1 + \tan x)^2}$$

(f) $D(x \tan x) = x \sec^2 x + \tan x$

(g) $$D\left(\frac{\sin x + \cos x}{\sin x - \cos x}\right)^2 = 2\left(\frac{\sin x + \cos x}{\sin x - \cos x}\right) D\left(\frac{\sin x + \cos x}{\sin x - \cos x}\right)$$

$$= \frac{-4(\sin x + \cos x)}{(\sin x - \cos x)^3}$$

12. Show that there are no points of the graph of $y = \sec x - \tan x$ at which the slope of the curve is zero.

$$f'(x) = \frac{\sin x - 1}{\cos^2 x}$$

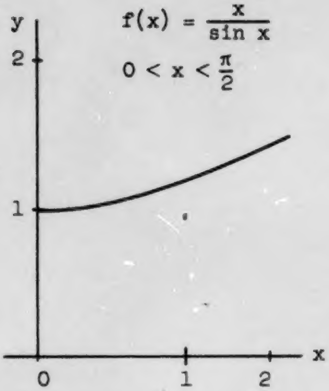
$\sin x = 1$ if and only if $\cos x = 0$, hence $f'(x)$ has no zeros.

13. Find all values of x for which the slope of the graph of $f(x) = \sin x \tan x$ is zero.

$$f'(x) = \sin x + \frac{\sin x}{\cos^2 x} = \sin x \left(1 + \frac{1}{\cos^2 x}\right) = 0 .$$

Thus, $x = k\pi$, k an integer.

14. (a) Sketch the graph of $f(x) = \frac{x}{\sin x}$ for $0 < x < \frac{\pi}{2}$.



(b) Examine $f'(x)$ and show that there is no value of x in the interval $0 < x < \frac{\pi}{2}$ for which $f'(x) = 0$.

$$f'(x) = \frac{\sin x - x \cos x}{\sin^2 x}$$

If $f'(x) = 0$, then $\sin x - x \cos x = 0$, or $\tan x = x$. But for all x , $0 < x < \frac{\pi}{2}$, $\tan x > x$; hence $f'(x) > 0$ for all x in the interval $0 < x < \frac{\pi}{2}$.

(c) Explain how your results support the fact that f is increasing on the given interval.

At this time a geometric argument should be presented. The analytic proof is given in Theorem 5-4a. Since f' is positive at every point on the interval, it would seem plausible that the graph of the function must slope in a positive direction as we proceed to the right.

15. (a) Find the maximum and minimum values of the function

$$f(x) = a \cos x + b \sin x$$

Use the identity

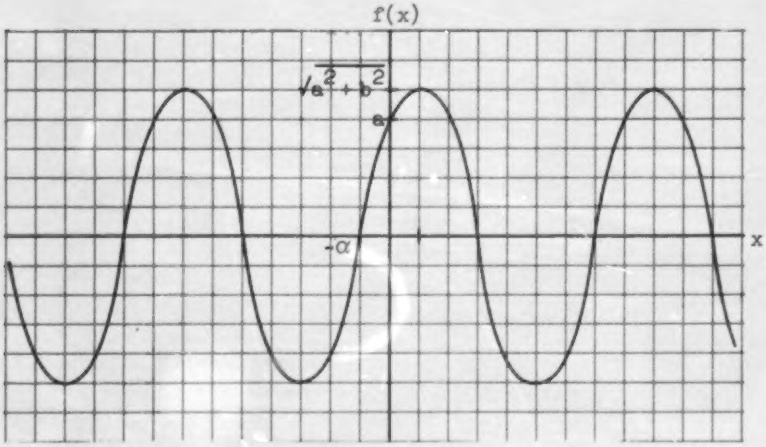
$$k \sin (\alpha + x) = k \sin \alpha \cos x + k \cos \alpha \sin x$$

and set $k \sin \alpha = a$, $k \cos \alpha = b$ so that $k^2 = a^2 + b^2$.

$$\therefore\ f(x) = k \sin \alpha \cos x + k \cos \alpha \sin x$$
$$= k \sin (\alpha + x) .$$

Thus, the maximum and minimum values of $f(x)$ are $\sqrt{a^2 + b^2}$ and $-\sqrt{a^2 + b^2}$, respectively.

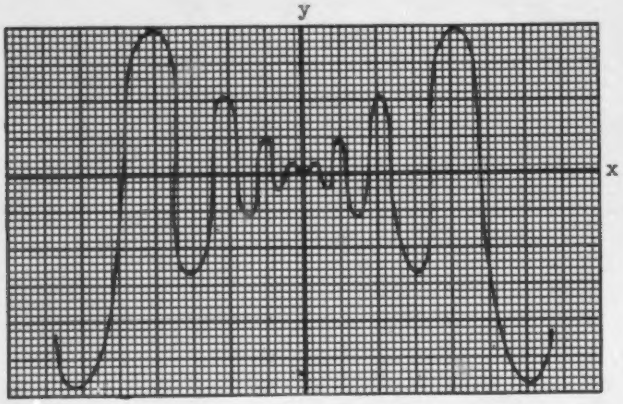
(b) Sketch the graph of f.



16. Consider $f : x \longrightarrow \sin \frac{1}{x}$ in the domain $0 < x \leq 1$.
Is it possible to define f at $x = 0$ such that the function is continuous in $[0,1]$?

No. Within any neighborhood of the origin, $\sin \frac{1}{x}$ takes on all values between -1 and 1. Thus, $\sin \frac{1}{x}$ has no limit at $x = 0$.

17. Consider $f : x \longrightarrow x \sin \frac{1}{x}$ for $x \neq 0$.
(a) Sketch the graph of f.



$f : x \rightarrow x \sin \frac{1}{x}$ for $x \neq 0$

(b) Is it possible to define f at $x = 0$ such that the function is continuous at $x = 0$?

Define
$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

$\lim\limits_{x \to 0} (x \sin \frac{1}{x}) = 0$, since if $\delta = \epsilon$, $\epsilon > |x| \geq |x \sin \frac{1}{x}|$.

Λ18. Does the function

$$f : x \longrightarrow \begin{cases} x^2 \sin \frac{\pi}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

have a derivative at $x = 0$?

Yes. $\frac{f(x+h) - f(x)}{h} = \frac{(0+h)^2 \sin \frac{\pi}{0+h} - 0}{h} = h \sin \frac{\pi}{h}$,

and $\lim_{h \to 0} h \sin \frac{\pi}{h} = 0$ so $f'(0) = 0$.

Note that the product rule does not apply here.

19. Given that the functions S and C satisfy the equations (differential equations)

$$DS = C \text{ and } DC = -S .$$

Show that $D(S^2 + C^2) = 0$.

$$\begin{aligned} D(S^2 + C^2) &= DS^2 + DC^2 = 2SDS + 2CDC \\ &= 2S(C) + 2C(-S) \\ &= 0 . \end{aligned}$$

20. Given that the functions f , g and h satisfy the equations

$$Df = g , \ Dg = h \text{ and } Dh = f .$$

Show that $D(f^3 + g^3 + h^3 - 3fgh) = 0$.

$$\begin{aligned} D(f^3 + g^3 + h^3 - 3fgh) &= 3f^2Df + 3g^2Dg + 3h^2Dh - 3ghDf - 3fhDg - 3fgDh \\ &= 3f^2g + 3g^2h + 3h^2f - 3g^2h - 3fh^2 - 3f^2g = 0 . \end{aligned}$$

TC4-5. Inverse Circular Functions.

For a discussion of inverse functions and related topics see Sections A2-3 and TC A2-3.

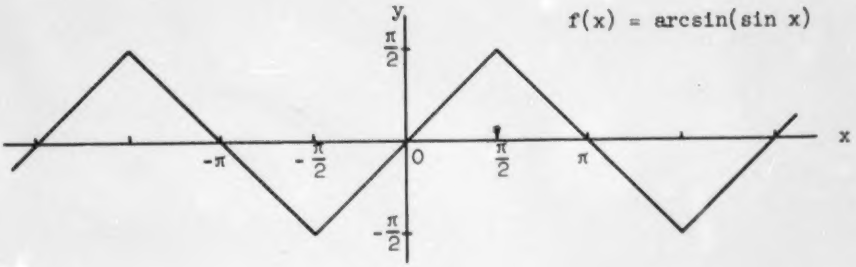
Solutions Exercises 4-5

1. Determine the domain and range and draw the graph of the function

(a) $f : x \longrightarrow \arcsin(\sin x)$

Domain: the set of all real numbers.

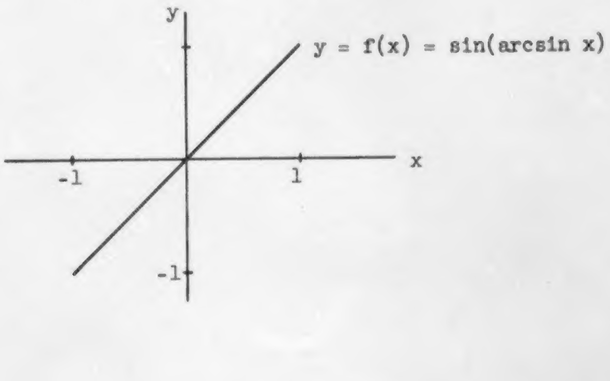
Range: the set of all real numbers $y = f(x)$ such that $-\frac{\pi}{2} \leq y \leq 2$.



(b) $f(x) = \sin(\arcsin x)$

Domain: all x such that $-1 \leq x \leq 1$.

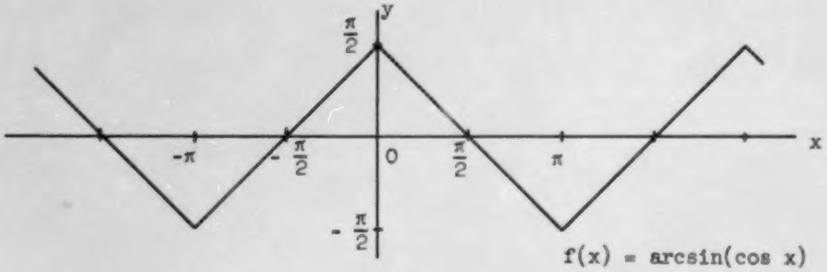
Range: all y such that $-1 \leq y \leq 1$.



(c) $f(x) = \arcsin(\cos x)$

Domain: set of all real numbers.

Range: all $y = f(x)$ where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.



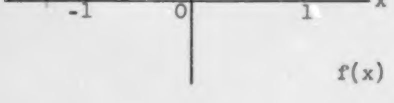
f(x) = arcsin(cos x)

(d) $f(x) = \cos(\arcsin x)$

Domain: all x where $-1 \leq x \leq 1$.

Range: all $y = f(x)$ where $0 \leq y \leq 1$.

The graph is the semicircle with center at origin, radius = 1 , and $y \geq 0$.

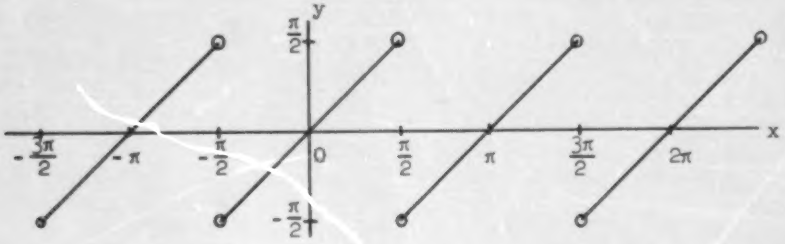


f(x) = cos(arcsin x)

(e) $f(x) = \arctan(\tan x)$

Domain: all real x except $x = (2n + 1)\frac{\pi}{2}$, n an integer.

Range: all $y = f(x)$ where $-\frac{\pi}{2} < y < \frac{\pi}{2}$.



2. Derive the formula

$$D \arccos x = \frac{-1}{\sqrt{1 - x^2}} ,$$

without using the identity which relates $\arccos x$ to $\arcsin x$.

Let $g : x \longrightarrow \arccos x$ and $f : y \longrightarrow \cos y$, where $y = \arccos x$, $-1 \leq x \leq 1$, $0 \leq y \leq \pi$.

Then $g'(x) = \frac{1}{f'(y)}$ (for $f'(y) \neq 0$, i.e., for $-\sin y \neq 0$)

for $0 < y < \pi$ and hence for $-1 < x < 1$.

$\therefore$ $D \arccos x = g'(x) = \frac{1}{-\sin y}$, where $\sin y > 0$,

$$= \frac{-1}{\sqrt{1 - x^2}} , \quad -1 < x < 1 .$$

3. Derive the formula

$$D \arctan x = \frac{1}{1 + x^2} .$$

Let $g : x \longrightarrow \arctan x$ and $f : y \longrightarrow \tan y$ where $y = \arctan x$ for all real x, and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

$$D \arctan x = g'(x) = \frac{1}{f'(y)}$$

$$= \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} , \text{ for } -\frac{\pi}{2} < y < \frac{\pi}{2} ,$$

$$= \frac{1}{1 + x^2} , \text{ for all } x .$$

4. Derive each of the following formulas:

(a) $D \operatorname{arccot} x = \dfrac{-1}{1 + x^2}$.

Let $g : x \longrightarrow \operatorname{arccot} x$ and $f : y \longrightarrow \cot y$ where $y = \operatorname{arccot} x$ for all real x and $0 < y < \pi$.

$$g'(x) = \frac{1}{f'(y)} = \frac{1}{-\csc^2 y} = \frac{-1}{1 + \cot^2 y} , \quad 0 < y < \pi ,$$

$$= \frac{-1}{1 + x^2} , \text{ for all } x .$$

(b) $D \operatorname{arcsec} x = \dfrac{1}{|x|\sqrt{x^2 - 1}}$.

Let $g : x \longrightarrow \operatorname{arcsec} x$ and $f : y \longrightarrow \sec y$ where $y = \operatorname{arcsec} x$, $|x| \geq 1$ and $0 \leq y < \frac{\pi}{2}$ or $\frac{\pi}{2} < y \leq \pi$.

Then $g'(x) = \dfrac{1}{f'(y)}$

$$= \frac{1}{\sec y \tan y} , \quad 0 < y < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < y < \pi$$

$$= \frac{1}{|x|\sqrt{x^2 - 1}} , \quad |x| > 1$$

Note: $\dfrac{1}{\sec y \tan y} = \dfrac{\cos^2 y}{\sin y} > 0$ for $0 < y < \frac{\pi}{2}$ or $\frac{\pi}{2} < y < \pi$, hence $g'(x) > 0$ for all x such that $|x| > 1$.

(c) $D \operatorname{arccsc} x = \dfrac{-1}{|x|\sqrt{x^2 - 1}}$.

Let $g(x) = \operatorname{arccsc} x$ and $f(y) = \csc y$ where $y = \operatorname{arccsc} x$ for $|x| \geq 1$ and $0 < |y| \leq \frac{\pi}{2}$.

$$\text{Then } g'(x) = \frac{1}{f'(y)}$$

$$= \frac{1}{-\csc y \cot y} , \quad 0 < |y| < \frac{\pi}{2}$$

$$= \frac{-1}{|x|\sqrt{x^2 - 1}} , \quad |x| > 1 .$$

Note: $g'(x) < 0$ since $\dfrac{-1}{\csc y \cot y} = \dfrac{-\sin^2 y}{\cos y} < 0$

for $0 < |y| < \frac{\pi}{2}$.

5. Evaluate:

(a) $D(\arcsin x + \arccos x) = \dfrac{1}{\sqrt{1 - x^2}} + \dfrac{-1}{\sqrt{1 - x^2}} = 0$

Note that $\arcsin x + \arccos x = \frac{\pi}{2}$ and $D(\frac{\pi}{2}) = 0$.

(b) $D(x^2 \arcsin x) = x^2 \cdot \dfrac{1}{\sqrt{1 - x^2}} + 2x \arcsin x$

$$= x(\frac{x}{\sqrt{1 - x^2}} + 2 \arcsin x)$$

(c) $D(\dfrac{x^2}{\arctan x}) = \dfrac{2x \arctan x - \dfrac{x^2}{1 + x^2}}{(\arctan x)^2}$

$$= \frac{2x(1 + x^2) \arctan x - x^2}{(1 + x^2)(\arctan x)^2}$$

(d) $D(\arcsin x)^3 = \dfrac{3(\arcsin x)^2}{\sqrt{1 - x^2}}$

(e) $D\left(\frac{1}{1 + \arcsin x}\right) = \frac{-1}{\sqrt{1 - x^2}\,(1 + \arcsin x)^2}$

6. Find $\lim\limits_{h \to 0} \frac{\arcsin h}{h}$ (Hint: What is the definition of the derivative of $f(x) = \arcsin x$ at $x = 0$?)

Let $f : x \longrightarrow \arcsin x$.

Then $\frac{f(x + h) - f(x)}{h}$, (for $x = 0$)

$$= \frac{\arcsin h - \arcsin 0}{h}$$

$$= \frac{\arcsin h}{h} .$$

$$\therefore \lim_{h \to 0} \frac{\arcsin h}{h} = f'(0) = \frac{1}{\sqrt{1 - 0^2}} = 1 .$$

7. (a) $D\left(\frac{\arcsin x}{1 + \arccos x}\right) = \frac{\frac{1 + \arccos x}{\sqrt{1 - x^2}} + \frac{\arcsin x}{\sqrt{1 - x^2}}}{(1 + \arccos x)^2}$

$$= \frac{1 + \frac{\pi}{2}}{\sqrt{1 - x^2}\,(1 + \arccos x)^2} .$$

(b) $D\left(\frac{\operatorname{arcsec} x}{1 - \operatorname{arccsc} x}\right) = \frac{1 - \operatorname{arccsc} x - \operatorname{arcsec} x}{|x|\sqrt{x^2 - 1}\,(1 - \operatorname{arccsc} x)^2} .$

(c) $D\left(\frac{1 - \arctan x}{1 + \arctan x}\right) = \frac{-2}{(1 + x^2)(1 + \arctan x)^2} .$

TC4-6. Compositions. Chain Rule.

In Section TC3-6 we supplied a geometric construction for the graph of a composite function. We now give a geometric interpretation of the chain rule. For the graphs of g , f , and gf we examine the slopes of appropriate chords and the derivative as a limit.

Let g and f be functions satisfying the hypothesis of Theorem 4-6. In Figure TC4-6a we show the graphs of g , f , gf , and two rectangles ($A_1B_1C_1D_1$ and $A_2B_2C_2D_2$) used in constructing the graph of gf (see Figures TC3-6c, 3-6d).

The slope of chord A_1A_2 joining points $(a, f(a))$ and $(x, f(x))$ on the graph of g is

$$\frac{f(x) - f(a)}{x - a}, \qquad x \neq a .$$

If $f(x) \neq f(a)$ the slope of chord C_1C_2 joining points $(f(a), g(f(a)))$ and $(f(x), g(f(x)))$ on the graph of g is

$$\frac{g(f(x)) - g(f(a))}{f(x) - f(a)}, \qquad x \neq a .$$

The slope of chord D_1D_2 joining points $(a, gf(a))$ and $(x, gf(x))$ on the graph of gf is

$$\frac{gf(x) - gf(a)}{x - a}, \qquad x \neq a .$$

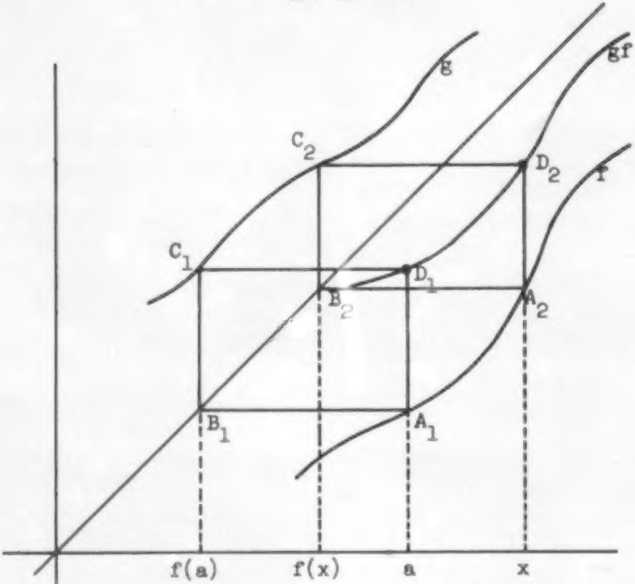


Figure TC4-6a

If $f(x) \neq f(a)$, then the product of the slopes of chords A_1A_2 and C_1C_2 is equal to the slope of chord D_1D_2; that is,

$$(1) \qquad \frac{f(x) - f(a)}{x - a} \cdot \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} = \frac{gf(x) - gf(a)}{x - a}, \qquad x \neq a .$$

If $f(x) = f(a)$, the slope of the chord A_1A_2 of the graph of f is zero, the slope of chord D_1D_2 of the graph of gf is zero, but points C_1 and C_2 coincide (Figure 4-6b). The slope of the graph of g at $x = f(a)$ is $g'(f(a))$, by hypothesis of Theorem 4-6. Thus, if $f(x) = f(a)$ the product of the slope of chord A_1A_2 and the slope of the graph of g at $x = f(a)$ is equal to the slope of chord D_1D_2; that is,

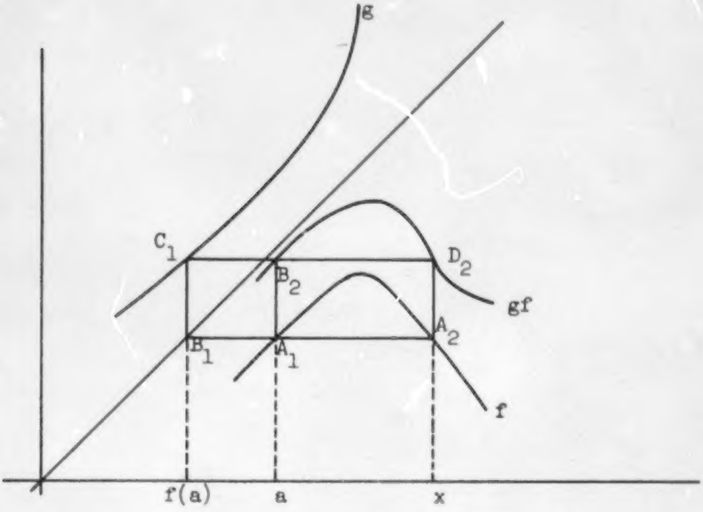


Figure TC4-6b

$$(2) \qquad \frac{f(x) - f(a)}{x - a} \cdot g'(f(a)) = \frac{gf(x) - gf(a)}{x - a}, \qquad x \neq a .$$

In either case, the limit as $f(x)$ approaches $f(a)$ of the slope of chord C_1C_2 is $g'(f(a))$; thus, it follows (by Theorems 3-6e and 3-4d) that the product of the slopes of the graphs of f at $x = a$ and g at $x = f(a)$ is equal to the slope of the graph of gf at $x = a$.

Solutions Exercises 4-6

1. For each of the following, find $Dfg(x)$, $Dgf(x)$, $Dff(x)$, and $Dgg(x)$.

(a) $f(x) = x^3 - 2x$, $g(x) = \sqrt{x}$.

(i) $Dfg(x) = D[x^{3/2} - 2x^{1/2}]$

$$= \frac{3}{2}x^{1/2} - x^{-1/2}$$

(ii) $Dgf(x) = D\sqrt{x^3 - 2x}$

$$= \frac{3x^2 - 2}{2\sqrt{x^3 - 2x}}$$

(iii) $Dff(x) = D[(x^3 - 2x)^3 - 2(x^3 - 2x)]$

$$= \big(3(x^3 - 2x)^2 - 2\big)(3x^2 - 2)$$

(iv) $Dgg(x) = Dx^{1/4}$

$$= \frac{1}{4}x^{-3/4}$$

(b) $f(x) = \sin x$, $g(x) = \cos x$.

(i) $Dfg(x) = D\sin(\cos x)$

$$= -\sin x\big(\cos(\cos x)\big)$$

(ii) $Dgf(x) = D\cos(\sin x)$

$$= -\cos x\big(\sin(\sin x)\big)$$

(iii) $Dff(x) = D\sin(\sin x)$

$$= \cos x\big(\cos(\sin x)\big)$$

(iv) $Dgg(x) = D\cos(\cos x)$

$$= \sin x\big(\sin(\cos x)\big)$$

(c) $f(x) = x^2$, $g(x) = \sin x$.

(i) $$Dfg(x) = D \sin^2 x$$
$$= 2 \sin x \cos x$$

(ii) $$Dgf(x) = D \sin(x^2)$$
$$= 2x \cos(x^2)$$

(iii) $$Dff(x) = Dx^4$$
$$= 4x^3$$

(iv) $$Dgg(x) = D \sin(\sin x)$$
$$= \cos x \cdot \cos(\sin x)$$

(d) $f(x) = \frac{x}{1 + x}$, $g(x) = \frac{x^2}{1 + x}$.

(i) $$Dfg(x) = \frac{1}{\left(1 + \frac{x^2}{1 + x}\right)^2} \cdot \frac{2x + x^2}{(1 + x)^2}$$
$$= \frac{2x + x^2}{(1 + x + x^2)^2}$$

(ii) $$Dgf(x) = \frac{2(\frac{x}{1 + x}) + (\frac{x}{1 + x})^2}{(1 + \frac{x}{1 + x})^2} \cdot \frac{1}{(1 + x)^2}$$
$$= \frac{2x(1 + x) + x^2}{(1 + 2x)^2(1 + x)^2}$$
$$= \frac{2x + 3x^2}{(1 + 2x)^2(1 + x)^2}$$

(iii) $$Dff(x) = D\left(\frac{f(x)}{1 + f(x)}\right) = \frac{f'(x)}{\left(1 + f(x)\right)^2}$$
$$= \frac{1}{(1 + 2x)^2}$$

(iv) $$Dgg(x) = D\left(\frac{g^2(x)}{1 + g(x)}\right)$$
$$= \frac{g(x)\left(2 + g(x)\right) g'(x)}{\left(1 + g(x)\right)^2}$$
$$= \frac{x^3(2 + x)(2 + 2x + x^2)}{(1 + x)^2(1 + x + x^2)^2} .$$

(e) $f(x) = \sin(x^2)$, $g(x) = \sqrt{1 - x^2}$.

(i) $$Dfg(x) = D\sin(1 - x^2)$$
$$= -2x\cos(1 - x^2)$$

(ii) $$Dgf(x) = D\sqrt{1 - \sin^2(x^2)}$$
$$= D|\cos(x^2)|$$
$$= \frac{-2x|\cos x^2|\sin x^2}{\cos x^2}$$

(iii) $$Dff(x) = D\sin\left(\sin^2(x^2)\right)$$
$$= 4x\sin(x^2)\cos(x^2)\cos\left(\sin^2(x^2)\right)$$

(iv) $$Dgg(x) = D\sqrt{1 - (\sqrt{1 - x^2})^2}$$
$$= D\sqrt{x^2} = D|x|$$
$$= \frac{|x|}{x}$$

2. Find $Dff(x)$ for

(a) $f(x) = \sin^2 x$.

$$Dff(x) = D\sin^2(\sin^2 x)$$
$$= \left(2\sin(\sin^2 x)\cos(\sin^2 x)\right)\ (2\sin x\cos x)$$
$$= \sin(2\sin^2 x)\sin 2x$$

(b) $f(x) = \tan^2 x$.

$$Dff(x) = D\tan^2(\tan^2 x)$$
$$= \left(2\tan(ten^2 x)\sec^2(\tan^2 x)\right)\ (2\tan x\cdot\sec^2 x)$$
$$= 4\tan x\sec^2 x\tan(\tan^2 x)\sec^2(\tan^2 x)$$

3. Find $f'(x)$ if $f(x)$ is:

(a) $(x^3 + 4)^{1/2}$.

$$D(x^3 + 4)^{1/2} = \frac{3x^2}{2(x^3 + 4)^{1/2}}$$

(b) $(2x^2 + 2)^{-1/2}$.

$$D(2x^2 + 2)^{-1/2} = \frac{-2x}{(2x^2 + 2)^{3/2}}$$

(c) $\dfrac{\sqrt{1-3x}+1}{\sqrt{1-3x}}$.

$$D\left(\frac{\sqrt{1-3x}+1}{\sqrt{1-3x}}\right) = \frac{3}{2(1-3x)^{3/2}}$$

(d) $\sqrt{\sin^2 x + x^2}$.

$$D\sqrt{\sin^2 x + x^2} = \frac{2x + \sin 2x}{2\sqrt{\sin^2 x + x^2}}$$

4. Evaluate:

(a) $D_x \sqrt{a^2 - x^2} = \dfrac{-x}{\sqrt{a^2 - x^2}}$.

(b) $D_x\left((x^2+1)^{1/2} + (x^2+1)^{-1/2}\right) = \dfrac{x^3}{(x^2+1)^{3/2}}$.

(c) $D_x\left(\dfrac{\sqrt{x^2-a^2}}{\sqrt{x^2+a^2}}\right) = \dfrac{2a^2x}{(x^2-a^2)^{1/2}(x^2+a^2)^{3/2}}$.

(d)
$$\begin{aligned} D_x\left(\frac{1+\sqrt{1-2x}}{\sqrt{1-2x}} - \sqrt{1-2x}\right) &= D\left(\frac{1}{\sqrt{1-2x}} + 1 - \sqrt{1-2x}\right) \\ &= \frac{1}{(1-2x)^{3/2}} + \frac{1}{(1-2x)^{1/2}} \\ &= \frac{2(1-x)}{(1-2x)^{3/2}} . \end{aligned}$$

(e)
$$\begin{aligned} D_x\left(x(2x^2-2x+1)^{-1/2}\right) &= (2x^2-2x+1)^{-1/2} + xD(2x^2-2x+1)^{-1/2} \\ &= \frac{1-x}{(2x^2-2x+1)^{3/2}} . \end{aligned}$$

5. Evaluate:

(a) $D_x(\sqrt{1+\cos x}) = \dfrac{D\cos x}{2\sqrt{1+\cos x}} = \dfrac{-\sin x}{2\sqrt{1+\cos x}}$.

(b) $D_x(x^2\sqrt{\sin x}) = \sqrt{\sin x}\, Dx^2 + x^2 D\sqrt{\sin x} = \dfrac{4x\sin x + x^2\cos x}{2\sqrt{\sin x}}$.

(c) $$D_x\Big(\cos\big(\cos(\cos x)\big)\Big) = -\sin\big(\cos(\cos x)\big)\ D\big(\cos(\cos x)\big)$$
$$= -\sin x \cdot \sin(\cos x) \cdot \sin\big(\cos(\cos x)\big) .$$

(d) $$D_x\big(\arcsin(\cos x)\big) = \frac{D \cos x}{\sqrt{1 - \cos^2 x}} = \frac{-\sin x}{|\sin x|} .$$

Note: If $0 < x < \pi$, $D[\arcsin(\cos x)] = -1$, and if $\pi < x < 2\pi$, $D[\arcsin(\cos x)] = 1$, etc. Refer to Exercises 4-5, Number 1c.

(e) $$D_x\big(\arctan(\arctan x)\big) = \frac{D \arctan x}{1 + (\arctan x)^2} = \frac{1}{(1 + x^2)\big(1 + (\arctan x)^2\big)} .$$

(f) $$D_x(x^2 \sin x \cos x) = \sin x \cos x\ Dx^2 + x^2 \cos x\ D \sin x + x^2 \sin x\ D \cos x$$
$$= x(x \cos 2x + \sin 2x) .$$

(g) $$D_x\left(\frac{\sin^2 x}{\sin(x^2)}\right) = \frac{\sin(x^2) D \sin^2 x - \sin^2 x\ D \sin(x^2)}{\sin^2(x^2)}$$
$$= \frac{\sin(x^2) \cdot \sin 2x - 2x \cdot \sin^2 x \cdot \cos(x^2)}{\sin^2(x^2)} .$$

(h) $$D_x\left(\tan\left(\frac{1 + x}{1 - x}\right)\right) = \sec^2\left(\frac{1 + x}{1 - x}\right) D\left(\frac{1 + x}{1 - x}\right)$$
$$= \frac{2}{(1 - x)^2} \sec^2\left(\frac{1 + x}{1 - x}\right) .$$

6. Evaluate:

(a) $$D_x\big(\arcsin(\sin x - \cos x)\big) = \frac{D(\sin x - \cos x)}{\sqrt{1 - (\sin x - \cos x)^2}}$$
$$= \frac{\sin x + \cos x}{\sqrt{\sin 2x}} .$$

(b) $$D_x \arcsin \frac{1 - x^2}{1 + x^2} = \frac{D\left(\frac{1 - x^2}{1 + x^2}\right)}{\sqrt{1 - \left(\frac{1 - x^2}{1 + x^2}\right)^2}}$$
$$= \frac{-2x}{|x|(1 + x^2)} .$$

(c) $$D_x\left(\arctan(x + \sqrt{1 + x^2})\right) = \frac{D\left(x + \sqrt{1 + x^2}\right)}{1 + \left(x + \sqrt{1 + x^2}\right)^2}$$

$$= \frac{x + \sqrt{1 + x^2}}{\sqrt{1 + x^2}\left(1 + (x + \sqrt{1 + x^2})^2\right)}$$

$$= \frac{1}{2(1 + x^2)} .$$

(d) $$D_x(\arctan \frac{1 + x}{1 - x}) = \frac{D(\frac{1 + x}{1 - x})}{1 + (\frac{1 + x}{1 - x})^2} = \frac{1}{1 + x^2} .$$

(e) $$D_x\left(\left(\arcsin(x^2)\right)^{-2}\right) = -2\left(\arcsin(x^2)\right)^{-3} D \arcsin(x^2)$$

$$= \frac{-4x}{\arcsin(x^2)^3 \sqrt{1 - x^4}} .$$

(f) $$D_x(\operatorname{arcsec} \sqrt{1 + x^2}) = \frac{D\sqrt{1 + x^2}}{|x|\sqrt{1 + x^2}} = \frac{x}{|x|(1 + x^2)} .$$

(g) $$D_x(\arctan \frac{x + 1}{x - 1} + \arctan x) = \frac{D\,\frac{x + 1}{x - 1}}{1 + (\frac{x + 1}{x - 1})^2} + \frac{1}{1 + x^2}$$

$$= 0 .$$

(h) $$D_x(\frac{\arcsin x}{\arctan x}) = \frac{\arctan x \; D \arcsin x - \arcsin x \; D \arctan x}{(\arctan x)^2}$$

$$= \frac{1}{\sqrt{1 - x^2}\,(\arctan x)} - \frac{\arcsin x}{(1 + x^2)(\arctan x)^2} .$$

(i) $$D_x\left(\arcsin(\arcsin x)\right) = \frac{D \arcsin x}{\sqrt{1 - (\arcsin x)^2}}$$

$$= \frac{1}{\sqrt{1 - x^2}\,\sqrt{1 - (\arcsin x)^2}} .$$

7. **Evaluate:**

(a) $D_v \sin x$, where $v = \cos x$.

Let $v = \cos x$. Then $\sin x = \sqrt{1 - v^2}$, if $\sin x > 0$, and $\sin x = -\sqrt{1 - v^2}$ if $\sin x < 0$. For $\sin x > 0$,

$$D_v \sin x = D_v \sqrt{1 - v^2} = \frac{-v}{\sqrt{1 - v^2}}$$

$$= \frac{-\cos x}{\sin x} = -\cot x \ .$$

For $\sin x < 0$,

$$D_v \sin x = D_v\left(-\sqrt{1 - v^2}\right) = \frac{v}{\sqrt{1 - v^2}}$$

$$= \frac{\cos x}{-\sin x} = -\cot x \ .$$

Hence for all x except $x = n\pi$, n an integer, $D_v \sin x = -\cot x$, where $v = \cos x$.

(b) $D_u \sqrt{1 - x^2}$, where $u = x^2$.

$D_u \sqrt{1 - x^2} = D_u \sqrt{1 - u}$, where $u = x^2$.

$$D_u \sqrt{1 - u} = \frac{-1}{2\sqrt{1 - u}} = \frac{-1}{2\sqrt{1 - x^2}} \ .$$

(c) $D_v(2 + 3 \cos^2 x)$, where $v = \sin x$.

$$\begin{aligned} D_v(2 + 3 \cos^2 x) &= D_v\left(2 + 3(1 - \sin^2 x)\right) \\ &= D_v\left(2 + 3(1 - v^2)\right) \\ &= D_v(5 - 3v^2) \\ &= -6v \\ &= -6 \sin x \ . \end{aligned}$$

8. Compute the limits of each of the following ratios.

(a) $$\lim_{x \to a} \frac{\sqrt{x^2 - 1} - \sqrt{a^2 - 1}}{x - a} = f'(a) \quad \text{where} \quad f(x) = \sqrt{x^2 - 1}$$

$$= \frac{a}{\sqrt{a^2 - 1}} .$$

(b) $$\lim_{x \to a} \frac{(\arccos x)^2 - (\arccos a)^2}{x - a} = g'(a)$$

where $g(x) = (\arccos x)^2$.

$$g'(a) = \left. \frac{-2 \arccos x}{\sqrt{1 - x^2}} \right|_{x = a}$$

$$= \frac{-2 \arccos a}{\sqrt{1 - a^2}}$$

9. If $f(x) = (Ax + B)\sin x + (Cx + D)\cos x$, determine the value of constants A , B , C , D such that for all x , $f'(x) = x \sin x$.

$f(x) = (Ax + B)\sin x + (Cx + D)\cos x$

$f'(x) = (Ax + B + C)\cos x + (A - Cx - D)\sin x$

$f'(0) = B + C = 0$, since $f'(0) = 0 \cdot \sin 0 = 0$.

$f'(\frac{\pi}{2}) = A - \frac{\pi}{2} C - D = \frac{\pi}{2}$, since $f'(\frac{\pi}{2}) = \frac{\pi}{2}$, and $A - D = \frac{\pi}{2}(1 + C)$.

$f'(\pi) = -\pi A = 0$, hence $A = 0$.

Hence $f'(x) = (-Cx - D)\sin x$ and $C = -1$, $D = 0$.

Thus $A = 0$, $B = 1$, $C = -1$, $D = 0$, and

$\therefore f(x) = \sin x - x \cos x$.

10. If $g(x) = (Ax^2 + Bx + C)\sin x + (Dx^2 + Ex + F)\cos x$, determine the value of constants A , B , C , D , E , F such that for all x , $g'(x) = x^2\cos x$.

$$g'(x) = \sin x(2Ax + B - Dx^2 - Ex - F) + \cos x(Ax^2 + Bx + C + 2Dx + E),$$

$$g'(0) = C + E = 0 ,$$

$$g'(\tfrac{\pi}{2}) = A\pi + B - D\frac{\pi^2}{4} - E\frac{\pi}{2} - F = 0 ,$$

$$g'(\tfrac{3\pi}{2}) = -(3A\pi + B - D\frac{9\pi^2}{4} - E\frac{3\pi}{2} - F) = 0 ,$$

$$g'(\tfrac{5\pi}{2}) = 5\pi A + B - \frac{25}{4}\pi^2 D - \frac{5\pi}{2}E - F = 0 .$$

$$\therefore -2A\pi + 2D\pi^2 + \pi E = 0$$

and $2A\pi - 4\pi^2 D - \pi E = 0$, whence

$D = 0$, $E = 2a$, $C = -2a$.

$$g'(\pi) = -(\pi^2 A + \pi B + C + 2\pi D + E) = -\pi^2 ,$$

$$g'(2\pi) = 4\pi^2 A + 2\pi B + C + 4\pi D + E = 4\pi^2 .$$

$\therefore 3\pi^2 A + \pi B + 2\pi D = 3\pi^2$, whence

$A = 1$, $B = 0$, $E = 2$, $C = -2$, $F = 0$ and

$$g(x) = (x^2 - 2)\sin x + 2x\cos x .$$

11. Determine the derivative $g'(x)$ in terms of $f'(x)$ if:

(a) $g(x) = f(x^3) + f(x^{-1/3})$.

$$g'(x) = 3x^2 f'(x^3) - \frac{1}{3}x^{-4/3} f'(x^{-1/3}) .$$

(b) $g(x) = f(\sin^2 x) + f(\cos^2 x)$.

$$\begin{aligned} g'(x) &= 2\sin x\cos x\, f'(\sin^2 x) - 2\cos x\sin x\, f'(\cos^2 x) \\ &= \sin 2x\, f'(\sin^2 x) - \sin 2x\, f'(\cos^2 x) \\ &= \sin 2x\big(f'(\sin^2 x) - f'(\cos^2 x)\big) . \end{aligned}$$

(c) $g(x) = f(\arcsin x) + f(\arctan x)$.

$$g'(x) = \frac{1}{\sqrt{1 - x^2}} f'(\arcsin x) + \frac{1}{1 + x^2} f'(\arctan x) .$$

12. Prove that the derivative of an even function is odd and vice verse (it is assumed that the derivative exists).

Let f be an even function, e.g., $f(x) = f(-x)$.

$Df(x) = Df(-x)$ or $f'(x) = -f'(-x)$ and f' is odd.

If g is an odd function, $g(x) = -g(-x)$.

Then

$Dg(x) = -Dg(-x)$ or $g'(x) = g'(-x)$ and g' is even.

These results also follow by considering the graphs of f for both cases.

13. Show that it is impossible to find polynomials p and q such that:

(a) $Dp(x) = \frac{1}{x}$. (b) $D\frac{1}{p(x)} = \frac{1}{x}$. (c) $D\frac{p(x)}{q(x)} = \frac{1}{x}$.

(a) Let n denote the degree of p $(n > 1)$ and assume $Dp(x) = \frac{1}{x}$.

Then, $xp'(x) = 1$. This is impossible as the degree of $xp'(x)$ is also n and the degree of 1 is zero.

(b) Assume $D\frac{1}{p(x)} = \frac{1}{x}$.

Then,
$$(p(x))^2 = -xp'(x) .$$

Equating the degrees of each side:

$$2m + 1 = m + n - 1 \quad \text{or} \quad m = n .$$

Also
$$2n = n \quad \text{or} \quad n = 0 .$$

This is also impossible.

∧(c) Assume $D\frac{p(x)}{q(x)} = \frac{1}{x}$ where the degree of p is n and the degree of q is m, where $m, n \neq 0$ and $p(x)$ and $q(x)$ are relatively prime (i.e., have no common factors).

Then,
$$x(q(x))^2 = q(x)p'(x) - p(x)q'(x) .$$

Equating the degrees of each side:

$$2m + 1 = m + n - 1 \quad \text{or} \quad m = n .$$

Also $$xq(x) - p'(x) = \frac{-p(x)q'(x)}{q(x)} .$$

Since the left-hand side is a polynomial, so must the right-hand side. Also, since p and q are relatively prime this implies that $q'(x)$ is divisible by $q(x)$. This is impossible since the degree of q' is one less than the degree of q .

TC4-7. Notation.

This section is included (at this point) as a reference: it need not be assigned as a unit for study in its own right.

Solutions Exercises 4-7

1. Let $y = \sin x$ and $x = t^2 + \frac{1}{t}$.

   Find $\left.\frac{dy}{dt}\right|_{t = 1}$ and $\left.\frac{dy}{dx}\right|_{x = 1}$.

$$\frac{dy}{dt} = D \sin (t^2 + \tfrac{1}{t}) = \cos (t^2 + \tfrac{1}{t})(2t - \tfrac{1}{t^2}) ,$$

$$\left.\frac{dy}{dt}\right|_{t = 1} = \cos 2 .$$

$$\frac{dy}{dx} = D \sin x = \cos x$$

$$\left.\frac{dy}{dx}\right|_{x = 1} = \cos 1 .$$

2. Let $y = f(x)$ and $x = h(t)$.

   Express $\left.\frac{dy}{dt}\right|_{t = t_0}$ in terms of t_0 .

$$\frac{dy}{dt} = Dfh(t) = f'(h(t)) (h'(t))$$

$$\left.\frac{dy}{dt}\right|_{t = t_0} = f'(h(t_0)) (h'(t_0)) .$$

3. Let $y = f(x)$, $x = h(t)$, $x_0 = h(t_0)$.

   Using Theorem 4-6 show that

$$\left.\frac{dy}{dx}\right|_{x = x_0} = \frac{\left.\frac{dy}{dt}\right|_{t=t_0}}{\left.\frac{dx}{dt}\right|_{t=t_0}} .$$

$$\frac{\left.\frac{dy}{dt}\right|_{t=t_0}}{\left.\frac{dx}{dt}\right|_{t=t_0}} = \frac{\left.\frac{d}{dt}\big(fh(t)\big)\right|_{t=t_0}}{\left.\frac{d}{dt}\big(h(t)\big)\right|_{t=t_0}} = \frac{f'\big(h(t)\big)\ h'(t)\big|_{t=t_0}}{h'(t)\big|_{t=t_0}}$$

$$= \frac{f'\big(h(t_0)\big)\ h'(t_0)}{h'(t_0)} = f'(x_0)$$

$$= \left.\frac{dy}{dx}\right|_{x = x_0} .$$

4. Find the following:

(a) $D_x \sin x\big|_{x=0} + D_x \sin x\big|_{x = \frac{\pi}{4}} = \cos 0 + \cos \frac{\pi}{4} = \frac{2 + \sqrt{2}}{2}$.

(b) $D_x(x^2 + \sin a \sin x)\Big|_{x = \frac{5\pi}{3}} = \frac{10\pi}{3} + \frac{1}{2} \sin a$.

(c) $\frac{d}{dx}(x^2 - a^2)\Big|_{x=a} = 2a$.

(d) $D_x\big(f(a) \sin x + f(x) \sin a + f(x) \sin x\big)\Big|_{x=a}$

$= \big(f(a)\cos x + \sin a\ f'(x) + \sin x\ f'(x) + f(x) \cos(x)\Big|_{x=a}$

$= 2\big(f(a)\cos a + f'(a) \sin a\big)$.

5. Let $y = f(t)$, $w = g(t)$, $t = h(x)$, $z = \frac{y}{w}$.

(a) Using Leibnizian notation, find $\frac{dz}{dx}$ in terms of $\frac{dy}{dt}$, $\frac{dw}{dt}$ and $\frac{dt}{dx}$.

$$\frac{dz}{dx} = \frac{d}{dx}\left(\frac{y}{w}\right) = \frac{w\,\frac{dy}{dx} - y\,\frac{dw}{dx}}{x^2}$$

$$= \frac{w\,\frac{dy}{dt} \cdot \frac{dt}{dx} - y\,\frac{dw}{dt}\,\frac{dt}{dx}}{w^2} .$$

(b) Using (a) express $\left.\frac{dz}{dx}\right|_{x=x_0}$ in terms of f' , g' , and h' .

$$\left.\frac{dz}{dx}\right|_{x=x_0} = \frac{gh(x_0)\ f'\big(h(x_0)\big)\,h'(x_0) - fh(x_0)g'\big(h(x_0)\big)\ h'(x_0)}{\big(gh(x_0)^{\ 2}\big)} .$$

TC4-8. Implicitly Defined Functions.

In elementary calculus texts it is customary to consider formal techniques of implicit differentiation and omit reference to questions that are involved. This text offers an honest, direct approach. It utilizes well chosen examples to point up the great convenience of implicit differentiation and indicate the impossibility of obtaining an explicit definition for all functions defined implicitly. The discussion of Equation (3) in Section 4-8.

$$x^2 \arctan z + z = \sin x$$

includes a plausible argument justifying the existence of the unique function $x \longrightarrow z$. This informal "proof" is in line with the formal proof of Theorem A5a.

Theorem A5b may be paraphrased: if an implicit relationship

$$F(x,y) = 0$$

defines a function $x \longrightarrow y$ or $y \longrightarrow x$, and if that function possesses a derivative when the point (x,y) satisfies the implicit relationship, then that derivative is generally obtainable by formal implicit differentiation. The text includes an example illustrating the case where the derivative at a point (x,y) exists but is not obtainable by implicit differentiation.

Solutions Exercises 4-8

1. For positive x , if $y = x^r$, where r is a rational number, say $r = \frac{p}{q}$ (p , q integers), then $y^q = x^p$. Assuming the existence of the derivative $D_x y$, derive the formula $D_x y = rx^{r-1}$ using implicit differentiation and the differentiation formula $D_x x^n = nx^{n-1}$, for integral n .

$y = x^r$, where $r = \frac{p}{q}$, $x > 0$. If $y^q = x^p$, $D_x y^q = D_x x^p$ and

$qy^{q-1}\, D_x y = px^{p-1}$, whence $D_x y = \frac{p}{q}\,\frac{x^{p-1}}{y^{q-1}} = \frac{p}{q}\,x^{\frac{p}{q}-1} = rx^{r-1}$.

2. For each of the following, find $D_x y$ without solving for y as a function of x.

   (a) $5x^2 + y^2 = 12$.

   $10x + 2y\,D_x y = 0$ and $D_x y = \frac{-5x}{y}$.

   (b) $2x^2 - y^2 + x - 4 = 0$.

   $4x - 2y\,D_x y + 1 = 0$

   $D_x y = \frac{4x + 1}{2y}$.

   (c) $y^2 - 3x^2 + 6y = 12$.

   $2y\,D_x y - 6x + 6\,D_x y = 0$

   $D_x y = \frac{3x}{y + 3}$.

   (d) $x^3 + y^3 - 2xy = 0$.

   $3x^2 + 3y^2\,D_x y - 2x\,D_x y - 2y = 0$

   $D_x y = \frac{2y - 3x^2}{3y^2 - 2x}$.

3. For each of the following use implicit differentiation to find $D_x y$

   (a) $x^2 = \frac{y - x}{y + x}$.

   $$2x = \frac{(y + x)(D_x y - 1) - (y - x)(D_x y + 1)}{(y + x)^2}$$

   $D_x y = \frac{x(y + x)^2 + y}{x}$.

   (b) $x^2 y + xy^2 = x^3$.

   $2xy + x^2 D_x y + y^2 + 2xy\,D_x y = 3x^2$

   $D_x y = \frac{3x^2 - 2xy - y^2}{x^2 + 2xy}$.

(c) $x^m y^n = 10$, (m , n integers).

$$mx^{m-1} y^n + nx^m y^{n-1} D_x y = 0$$

$$D_x y = \frac{-my}{nx} \ .$$

(d) $\sqrt{xy} + x = y^{-1}$.

$$\frac{1}{2}(xy)^{-1/2}(x\ Dy + y) + 1 = -\frac{Dy}{y^2}$$

$$D_x y = \frac{-2y^2\sqrt{xy} - y^3}{xy^2 + 2\sqrt{xy}} \ .$$

4. Use implicit differentiation to find $D_y x$

(a) $x\sqrt{y} + y\sqrt{x} = a\sqrt{a}$, a constant.

$$\sqrt{y}\ Dx + \frac{x}{2\sqrt{y}} + \frac{y}{2\sqrt{x}}\ Dx + \sqrt{x} = 0$$

$$D_y x = \frac{-x(\sqrt{x} + 2\sqrt{y})}{y(2\sqrt{x} + \sqrt{y})} \ .$$

(b) $2x^2 + 3xy + y^2 + x - 2y + 1 = 0$.

$$4xDx + 3yDx + 3x + 2y + Dx - 2 = 0$$

$$D_y x = \frac{2 - 3x - 2y}{4x + 3y + 1} \ .$$

(c) $(x + y)^{1/2} + (x - y)^{1/2} = 4$.

$$\frac{1}{2(x + y)^{1/2}}(Dx + 1) + \frac{1}{2(x - y)^{1/2}}(Dx - 1) = 0$$

$$D_y x = \frac{\sqrt{x + y} - \sqrt{x - y}}{\sqrt{x + y} + \sqrt{x - y}} \ .$$

(d) $3x^2 + x^2y^2 = y^4 + 5$.

$$6x\mathrm{D}x + 2xy^2\mathrm{D}x + 2x^2y = 4y^3$$

$$\mathrm{D}_y x = \frac{2y^3 - x^2y}{3x + xy^2} .$$

(e) $4x^2 + 3xy - 7y^2 = 0$.

$$8x\mathrm{D}x + 3y\mathrm{D}x + 3x - 14y = 0$$

$$\mathrm{D}_y x = \frac{14y - 3x}{8x + 3y} .$$

5. For each equation, find the slope of the curve represented at the stated point.

(a) $2x^2 + 3xy + y^2 + x - 2y + 1 = 0$ at the point $(-2,1)$.

$$4x + 3y + 3x\,\mathrm{D}_x y + 2y\,\mathrm{D}_x y + 1 - 2\mathrm{D}_x y = 0$$

At $(-2,1)$, $\mathrm{D}_x y = -\frac{2}{3}$.

(b) $x^3 + y^2x^2 + y^3 - 1 = 0$ at the point $(1,-1)$.

$$3x^2 + 2yx^2\,\mathrm{D}_x y + 2xy^2 + 3y^2\,\mathrm{D}_x y = 0$$

At $(1,-1)$, $\mathrm{D}_x y = -5$.

(c) $x^2 - x\sqrt{xy} - 6y^2 = 2$ at the point $(4,1)$.

$$2x - \frac{3}{2}\sqrt{xy} - \frac{1}{2}x^{3/2}y^{-1/2}\,\mathrm{D}_x y - 12y\,\mathrm{D}_x y = 0$$

At $(4,1)$, $\mathrm{D}_x y = \frac{5}{16}$.

(d) $x \cos y = 3x^2 - 5$ at the point $(\sqrt{2}, \frac{\pi}{4})$.

$$x(-\sin y\,\mathrm{D}_x y) + \cos y = 6x$$

At $(\sqrt{2}, \frac{\pi}{4})$, $\mathrm{D}_x y = -\frac{11\sqrt{2}}{2}$.

6. For each equation, find the slope of the curve represented at the point or points where $x = y$. Give a geometric explanation for these results.
   (a) $x^3 - 3axy + y^3 = 0$.

$$3x^2 - 3ax\, D_x y - 3ay + 3y^2\, D_x y = 0$$

$$D_x y\Big|_{x=y} = \frac{-3x^2 + 3ay}{-3ax + 3y^2}\Bigg|_{x=y\neq 0} = -1 .$$

   (b) $x^m + y^m = 2$.

$$mx^{m-1} + my^{m-1}\, D_x y = 0$$

$$D_x y\Big|_{x=y} = \frac{-mx^{m-1}}{my^{m-1}}\Bigg|_{x=y\neq 0} = -1 .$$

   (c) $x^2 + y^2 = 2axy + a^2$. $(a \neq 0)$

$$2x + 2y\, D_x y = 2ax\, D_x y + 2ay$$

$$D_x y\Big|_{x=y} = \frac{2ay - 2x}{2y - 2ax}\Bigg|_{x=y} = -1 .$$

All three curves are symmetric about the line $y = x$. Thus at the point where $x = y$ the tangents to the curves are orthogonal to the line (Refer to Exercises 2-5, No. 10).

7. Find $D_x y$ by implicit differentiation.
   (a) $a \sin y + b \cos x = 0$ (a, b constant).

$$a \cos y\, D_x y - b \sin x = 0$$

$$D_x y = \frac{b \sin x}{a \cos y} .$$

(b) $x \cos y + y \sin x = 0$.

$$\cos y - x \sin y \, D_x y + \sin x \, D_x y + y \cos x = 0$$

$$D_x y = \frac{\cos y + y \cos x}{x \sin y - \sin x} .$$

(c) $\sin xy = \sin x + \sin y$.

$$(\cos xy)(y + xD_x y) = \cos x + \cos y \, D_x y$$

$$D_x y = \frac{y \cos xy - \cos x}{\cos y - x \cos xy} .$$

(d) $\csc(x + y) = y$.

$$-\csc(x + y) \cot(x + y)(1 + D_x y) = D_x y$$

$$D_x y = - \frac{\csc(x + y) \cot(x + y)}{\csc(x + y) \cot(x + y) + 1} .$$

(e) $x \tan y - y \tan x = 1$.

$$\tan y + x \sec^2 y \, D_x y - \tan x \, D_x y - y \sec^2 x = 0$$

$$D_x y = \frac{\tan y - y \sec^2 x}{\tan x - x \sec^2 y} .$$

(f) $\tan xy - x^2 = 0$.

$$(\sec^2 xy)(y + xD_x y) - 2x = 0$$

$$D_x y = \frac{2x - y \sec^2 xy}{x \sec^2 xy} ,$$

(g) $y \sin x = x \tan y$,

$$\sin x \, D_x y + y \cos x = \tan y + x \sec^2 y \, D_x y$$

$$D_x y = \frac{\tan y - y \cos x}{\sin x - x \sec^2 y} .$$

8. If $0 < x < a$, then the equation $\sqrt{x} + \sqrt{y} = \sqrt{a}$ defines y as a function of x . Assuming the existence of the derivative show without solving for y that $f'(x)$ is always negative.

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} D_x y = 0$$

$D_x y = - \dfrac{\sqrt{y}}{\sqrt{x}}$ which is always negative since $x , y > 0$.

9. Assuming that $D_x y = D_y x = 0$ (i.e., x and y are independent), find the following:

(a) $D_x(x^2 + xy + \cos y)$.

$$D_x(x^2 + xy + \cos y) = 2x + x\, D_x y + y - \sin y\, D_x y = 2x + y .$$

(b) $D_x(y^2) + D_y(x^2)$.

$$D_x(y^2) + D_y(x^2) = 2y\, D_x y + 2x\, D_y x = 0 .$$

(c) $D_x(x^2) + D_y(y^2)$.

$$D_x(x^2) + D_y(y^2) = 2x + 2y .$$

(d) $D_x f(xy) + D_y f(x)$.

$$\begin{aligned} D_x f(xy) + D_y f(x) &= f'(xy)(x\, D_x y + y) + f'(x)\, D_y x \\ &= yf'(xy) . \end{aligned}$$

(e) $D_x(xy)^2$.

$$D_x(xy)^2 = 2(xy)(x\, D_x y + y) = 2xy^2 .$$

10. Let c_1 and c_2 be two curves which intersect at the point (x_0,y_0) and let the slopes of c_1 and c_2 at (x_0,y_0) be m_1 and m_2, respectively. If the product m_1m_2 equals -1, we say that the curves c_1 and c_2 are orthogonal.

(a) Show that the lines with equations $4y - 3x - 40 = 0$ and $3y + 4x + 15 = 0$ are orthogonal.

$4D_xy - 3 = 0$ and $D_xy = \frac{3}{4} = m_1$

$3D_xy + 4 = 0$ and $D_xy = -\frac{4}{3} = m_2$;

$m_1 \cdot m_2 = -1$ hence the lines are orthogonal.

(b) Show that the circle $x^2 + y^2 = r^2$, r constant, is orthogonal to the line $y = mx$, m constant.

$2x + 2y\,D_xy = 0$ and $D_xy = -\frac{x}{y} = m_1$

$m_2 = m = \frac{y}{x}$

$m_1m_2 = -1$.

11. Find the points of intersection of the ellipse $x^2 + 10y^2 = 10$ and the hyperbola $x^2 - 8y^2 = 8$, and the slopes of the curves at these points of intersection. Show that the curves are orthogonal.

Points of intersection are

$\left(\frac{4\sqrt{5}}{3}, \frac{1}{3}\right)$, $\left(\frac{4\sqrt{5}}{3}, -\frac{1}{3}\right)$, $\left(\frac{-4\sqrt{5}}{3}, \frac{1}{3}\right)$ and $\left(\frac{-4\sqrt{5}}{3}, -\frac{1}{3}\right)$.

For the ellipse $D_xy = m_1 = -\frac{x}{10y}$

For the hyperbola $D_xy = m_2 = \frac{x}{8y}$

$m_1m_2 = \frac{-x^2}{80y^2}$. For each point of intersection, $m_1 \cdot m_2 = -1$.

12. Show that the family of curves $y^2 = 4a(x + a)$ is self-orthogonal; i.e., each two members of the family, $y^2 = 4a_1(x + a_1)$ and $y^2 = 4a_2(x + a_2)$, that intersect, necessarily intersect at right angles.

Let (x_0,y_0) be the point of intersection of the two members of the family. The slopes of the curves $y^2 = 4a_1(x + a_1)$ and $y^2 = 4a_2(x + a_2)$ at (x_0,y_0) are $\frac{2a_1}{y_0}$ and $\frac{2a_2}{y_0}$, respectively. At (x_0,y_0) we have $a_1(x_0 + a_1) = a_2(x_0 + a_2)$ whence $x_0 = -(a_1 + a_2)$ and $y_0^2 = 4a_1(x_0 + a_1) = -4a_1a_2$. Thus, the product of the slopes of the curves at (x_0,y_0) is $\frac{2a_1}{y_0} \cdot \frac{2a_2}{y_0} = \frac{4a_1a_2}{-4a_1a_2} = -1$. The result is immediate.

Λ13. For what values of k will there be exactly one line passing through the point $(0,k)$ and orthogonal to the parabola $y = x^2$? For what values of k **will there be exactly three orthogonal lines?**

Let the equation of the line through $(0,k)$ be

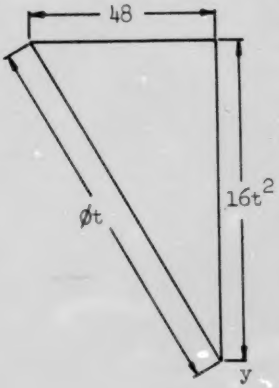
$$y = mx + k .$$

The line intersects $y = x^2$ at $x = \frac{m \pm \sqrt{m^2 + 4k}}{2}$. Slope of parabola at points of intersection is $m \pm \sqrt{m^2 + 4k}$. For orthogonality $m(m \pm \sqrt{m^2 + 4k}) = -1$ and $k = \frac{1}{2} + \frac{1}{4m^2}$. Hence, for all real numbers m $k > \frac{1}{2}$. Thus, there will be two orthogonal lines of the form $y = mx + k$ which pass through the point $(0,k)$ and are orthogonal to the parabola. However, since $D_xy = 2x$ is equal to zero at $(0,0)$ the y-axis is orthogonal to the parabola at the origin.

If $k \leq \frac{1}{2}$, only the y-axis is orthogonal to parabola.

If $k > \frac{1}{2}$, 3 lines are orthogonal to parabola.

14. A ball dropped out of a window falls $16t^2$ feet in t seconds. An observer is watching from another window at the same height 48 feet away. At what rate is the distance of the ball from the observer increasing two seconds after the ball is dropped?



(a) Write an equation which implicitly defines the distance $y = \phi(t)$ between the observer and the ball at time t.

$$y^2 = 48^2 + (16t^2)^2.$$

(b) Use implicit differentiation to answer the question of the problem.

$$2y\,D_t y = 4 \cdot 16^2 \cdot t^3$$

At $t = 2$, $D_t y = \frac{256}{5}$

The distance of the ball from the observer is increasing at the rate of $\frac{256}{5}$ ft/sec. two seconds after the ball is dropped.

15. (a) Given the simple harmonic motion is described by the function $\rho : t \longrightarrow \sin(\omega t + c)$ where ω and c are constants. Find the velocity at time $t = t_0$.

$$v = D_t \sin(\omega t + c) = \omega \cos(\omega t + c)$$

$$D_t \sin(\omega t + c)\Big|_{t=t_0} = \omega \cos(\omega t_0 + c).$$

(b) Simple harmonic motion may also be described by the function $\rho : t \longrightarrow \cos(\omega t + c)$ where ω and c are constants. Find the velocity at time $t = t_0$.

$$v = D_t \cos(\omega t + c) = -\omega \sin(\omega t + c)$$

$$D_t \cos(\omega t + c)\Big|_{t=t_0} = -\omega \sin(\omega t_0 + c).$$

(c) In what sense are the motions in (a) and (b) the same?

The magnitudes of the motion described in (a) and (b) are the same except for a phase shift of $\frac{\pi}{2}$ radians.

16. If a simple harmonic motion is described by the function $\rho : t \longrightarrow A \sin \omega t + B \cos \omega t$ where A, B and ω are constants, determine the maximum speed.

(Refer to Solutions Exercises 4-4, No. 15.)

$A \sin \omega t + B \cos \omega t = \sqrt{A^2 + B^2} \sin(\omega t + \alpha)$

$v : t \longrightarrow \omega \sqrt{A^2 + B^2} \cos(\omega t + \alpha)$

Speed is maximum when $|\cos(\omega t + \alpha)| = 1$

Maximum speed $= \omega \sqrt{A^2 + B^2}$.

Solutions Miscellaneous Exercises

1. Evaluate

(a) $D(x + \frac{1}{x})^{1/2} = \frac{1}{2}(x + \frac{1}{x})^{-1/2}\left(1 - \frac{1}{x^2}\right)$.

(b) $D(\arcsin \sqrt{x})^2 = \dfrac{\arcsin \sqrt{x}}{\sqrt{x}\sqrt{1 - x}}$.

(c) $D\sqrt{3x^2 - 1}\Big|_{x = \frac{3}{2}} = \dfrac{3x}{\sqrt{3x^2 - 1}}\Bigg|_{x = \frac{3}{2}} = \dfrac{9\sqrt{23}}{23}$.

(d) $D_u\sqrt{x - x^2}$, where $u = x^2$ and $0 < x < 1$.

$$D_u\sqrt{x - x^2} = D_x\sqrt{x - x^2} \cdot D_u x$$

$$= \frac{1 - 2x}{2\sqrt{x - x^2}} \cdot \frac{1}{2x} = \frac{1 - 2x}{4x\sqrt{x - x^2}} .$$

(e) $D\sqrt{x + \sqrt{x}} = \dfrac{1}{2\sqrt{x + \sqrt{x}}} \left(1 + \dfrac{1}{2\sqrt{x}}\right)$.

(f) $D(x^3 - 2)^6 = 18x^2(x^3 - 2)^5$.

(g) $D\sqrt[3]{x^2 - 3} = \dfrac{2x}{3(x^2 - 3)^{2/3}}$.

(h) $D_u(x^2 - x^{-1/2} + x^{-2})$, where $u = \sqrt{x}$.

$$\begin{aligned} D_u(x^2 - x^{-1/2} + x^{-2}) &= D_x(x^2 - x^{-1/2} + x^{-2})D_u x \\ &= (2x + \tfrac{1}{2}x^{-3/2} - 2x^{-3})(2x^{1/2}) \\ &= 4x^{3/2} + x^{-1/2} - 4x^{-5/2} . \end{aligned}$$

(i) $D\left(\dfrac{x}{x + \sqrt{a^2 - x^2}}\right) = \dfrac{a^2}{(x + \sqrt{a^2 - x^2})^2\sqrt{a^2 - x^2}}$.

(j) $D\sqrt{x + 1} = \dfrac{1}{2\sqrt{x + 1}}$.

(k) $D_v(\sin x \cdot \cot x)$, where $v = \cos x$.

$$\begin{aligned} D_v(\sin x \cdot \cot x) &= D_x(\sin x \cdot \cot x) \cdot D_v x \\ &= (-\sin x)\left(\frac{-1}{\sin x}\right) = 1 . \end{aligned}$$

(ℓ) $D(\sin x^{-1/2} - \cos x^{1/2}) = \dfrac{-\cos(x^{-1/2})}{2x^{3/2}} + \dfrac{\sin(x^{1/2})}{2x^{1/2}}$.

(m) $D_v\left(\dfrac{1 - \sin x}{1 + \cos x}\right)$ where $v = \cos \dfrac{x}{2}$.

$$\begin{aligned} D_v\left(\frac{1 - \sin x}{1 + \cos x}\right) &= D_x\left(\frac{1 - \sin x}{1 + \cos x}\right) \cdot D_v x \\ &= \frac{\sin x}{(1 + \cos x)^2} \cdot \frac{-2}{\sin \frac{x}{2}} = \frac{-2 \sin x}{\sin \frac{x}{2}(1 + \cos x)^2} . \end{aligned}$$

(n) $D\left(\dfrac{2 + t}{3t^2 - 1}\right) = \dfrac{-3t^2 - 12t - 1}{(3t^2 - 1)^2}$.

(o) $D\,\dfrac{(2x - 3)^2}{x^3 - 1} = \dfrac{4(x^3 - 1)(2x - 3) - 3x^2(2x - 3)^2}{(x^3 - 1)^2}$.

(p) $D(2x^3 + 5x^2 - x + 2)^{10} = 10(6x^2 + 10x - 1)(2x^3 + 5x^2 - x + 2)^9$.

(q) $D_v\left(\frac{1 - x}{x^2}\right)$ where $v = \arcsin x$.

$$D_v(\frac{1 - x}{x^2}) = D_x(\frac{1 - x}{x^2})D_v x$$
$$= (\frac{x - 2}{x^3})\sqrt{1 - x^2} .$$

(r) $D \cos\sqrt{1 - x^2} = \frac{x \sin\sqrt{1 - x^2}}{\sqrt{1 - x^2}}$.

(s) $D(\arcsin(x^2) + \arccos(\frac{\pi}{2} - x^2) = \frac{2x}{\sqrt{1 - x^4}} + \frac{2x}{\sqrt{1 - (\frac{\pi}{2} - x^2)^2}}$.

(t) $D(\tan (x^{-1} - x) = -\left(\frac{1}{x^2} + 1\right) \sec^2(x^{-1} - x)$.

2. In Section 2-4 we defined the velocity of an object whose location on a straight line at time $t = t_0$ is given by $s = \phi(t)$ as the limit of the ratio

$$\frac{\phi(t) - \phi(t_0)}{t - t_0}$$

and in Section 2-5 we observed that this limit is the value of the derivative ϕ' at $t = t_0$. Experimentally it has been established that the distance covered in time t by a freely falling body is proportional to t^2 , and therefore it can be represented by the function $\phi : t \longrightarrow ct^2$, where c is a positive constant. Show that the velocity of a freely falling body is directly proportional to the time.

$$\phi : t \longrightarrow ct^2$$
$$\phi'(t) = 2ct$$
$$= kt \text{ , where } k = 2c \text{ is a positive constant.}$$

3. Suppose a projectile is ejected at a point P which is 20 feet above the ground with initial velocity of v_0 feet per second. Neglect friction and assume that the projectile moves up and down in a straight line. Let $\theta(t)$ denote the height (above P) in feet that the projectile attains t seconds after ejection. Note that if gravitational attraction were not acting on the projectile, it would continue to move upward with a constant velocity, traveling a distance of v_0 feet per second, so that its height at time t would be given by $\theta(t) = v_0 t$. We know that the force of gravity acting on the projectile causes it to slow down until its velocity is zero and then travel back to the earth. On the basis of physical ex-

periments the formula $\theta(t) = v_0 t - \frac{g}{2}t^2$, where g represents the force of gravity, is used to represent the height (above P) of the projectile as long as it is aloft. Note that $\theta(t) = 0$ when $t = 0$ and when $t = \frac{2v_0}{g}$. This means that the projectile returns to the intial 20 foot level after $\frac{2v_0}{g}$ seconds.

(a) Find the velocity of the projectile at $t = t_0$ (in terms of v_0 and g).

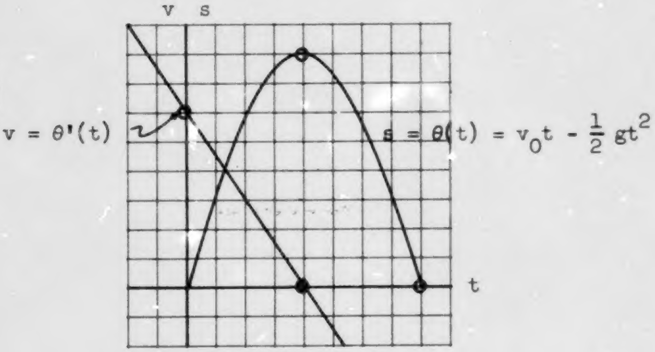
$$\theta(t) = v_0 t - \frac{1}{2} g t^2 .$$

The velocity is given by

$$\theta'(t) = v_0 - gt$$

whence $$\theta'(t_0) = v_0 - g t_0 .$$

(b) Sketch the s vs. t and the v vs. t curves on the same set of axes.



(c) Compute (in terms of v_0) the time required for the velocity to drop to zero.

$$\phi'(t) = v_0 - gt = 0 \text{ if } t = \frac{v_0}{g} .$$

(d) What is the velocity on return to the initial 20 foot level?

$$\phi(t) = t(v_0 - \frac{1}{2} gt) = 0 \text{ if } t = 0 \text{ or } t = \frac{2v_0}{g} .$$

The velocity on return to the initial 20 foot level is given by

$$\phi'\left(\frac{2v_0}{g}\right) = - v_0 .$$

(e) Assume that the projectile returns to earth at a point 30 feet below the initial take off point P . What is the velocity at impact?

We know that $(v_{final})^2 - (v_0)^2 = 2gs$

since $v_f^2 = (v_0 + gt)^2$.

Since the object is traveling downward, g is positive. Thus $(v_f)^2 = v_0^2 + 2gs$ and $v_f^2 = v_0^2 + 60g$ since $s = 30$.

Hence, $v_f = \sqrt{v_0^2 + 60g}$.

Teacher's Commentary

Chapter 5

APPLICATIONS OF THE DERIVATIVE

TC5-1. Introduction.

In Chapter 5 we study the derivative f' in order to obtain information about the function f. We here note some generalizations about functions which are derivatives.

1. The derivative of a continuous function is not necessarily continuous.

Example TC5-1. Consider the function f given by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

The derivative, given by

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0, \end{cases}$$

is continuous for all $x \neq 0$. The derivative cannot be continuous at 0 since $\lim_{x \to 0} f'(x)$ does not exist.

2. Any derivative f' which exists throughout an interval has the intermediate value property; that is, if f' takes on any two values in the interval $[a,b]$, then it takes on every value between them (Exercises 5-3, No. 20). This property is quite remarkable in that it holds for derivatives having points of discontinuity (Example TC5-1).

3. An immediate consequence of the intermediate value property of derivatives is the fact that a derivative, if discontinuous, cannot have a finite jump. Note that in Example TC5-1 neither $\lim_{x \to 0^-} f'(x)$ not $\lim_{x \to 0^+} f'(x)$ exists. (See Exercises 3-4, No. 16 for comment on left-sided and right-sided limits.)

Solutions Exercises 5-1

1. (a) Obtain an expression for the first and higher derivatives of x^n where n is a natural number.

$$Dx^n = nx^{n-1} \text{ , if } n > 0 .$$

$$Dnx^{n-1} = nDx^{n-1}$$

$$= n(n - 1)x^{n-2} \text{ , if } n > 1 .$$

In general, if $n \geq i$,

$$D^i x^n = n(n - 1)(n - 2) \dots (n - i + 1)x^{n-i} ,$$

which can formally be proved by mathematical induction (Section A3-1). For the case $i > n$, $D^i x^n = 0$, since $D^n x^n = n(n - 1)(n - 2) \dots (2)(1) = n!$, and $Dn! = 0$. Higher order derivatives are then all zero.

(b) Do the same for $x^{p/q}$ where $\frac{p}{q}$ is rational and not integral, p and q relatively prime. What is the domain of f' when $f(x)=x^{1/2}$? For what values of $\frac{p}{q}$ is the domain of f' different from that of $f : x \rightarrow x^{p/q}$? Answer the same question for higher derivatives of $x^{p/q}$.

By corollary to Theorem 4-3,

$$D_x x^r = rx^{r-1}$$

for all rational numbers r . By induction, just as in 1(a) above,

$$D^n x^r = r(r - 1)(r - 2) \dots (r - n + 1)x^{r-n}$$

since $r - n$ is rational and $r - n \neq 0$.
If $0 < \frac{p}{q} < 1$, then the domains of f and f' differ: domain f includes zero, and domain f' does not ($f'(x)$ will have x in the denominator and $f(x)$ will not). In general, the domain of $f^{(n)}$ will differ from that of f if $0 < \frac{p}{q} < n$.

Note. p and q are relatively prime. If $\frac{p}{q}$ were integral, Exercise 1(a) above would apply.

2. What is the twenty-third derivative of

(a) $\sin x$?

(b) $\cos x$?

(c) $\sin 2x$?

(d) $5x^{17} = \sqrt{2}\, x^{10} - 83x^3 - 70425$?

(a)
$$\begin{aligned} f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x \\ f^{IV}(x) &= \sin x \end{aligned}$$

We see that successive differentiation is a cycle with period four. Thus,

$$\begin{aligned} f^{(25)}(x) &= f^{(4\cdot5+3)}(x) \\ &= f^{(3)}(x) \\ &= -\cos x \,. \end{aligned}$$

(b) Since $\cos x = D \sin x$,

$$\begin{aligned} D^{23}\cos x &= D^{23}D \sin x \\ &= D^{24}\sin x \\ &= \sin x \,, \end{aligned}$$

by the same argument as in 2(a).

(c) Since $D_x \sin 2x = 2 \cos 2x$,

$$\begin{aligned} D^{23}\sin 2x &= D^{4\cdot5+3}\sin 2x \\ &= -\, 2^{23}\cos 2x \,, \end{aligned}$$

(d)
$$f^{(23)}(x) = 0 \,.$$

See Exercise No. 1(a).

The result of problem 3g is used in 3h; hence, if 3h is assigned, 3g should also be assigned.

3. Find the n-th derivatives of the following functions.

(a) $f : x \rightarrow (ax + b)^{-1}$

$$f'(x) = \frac{-a}{(ax + b)^2} ,$$

$$f''(x) = \frac{2a^2}{(ax + b)^2} ,$$

and, in general,

$$f^{(n)}(x) = \frac{(-1)^n n! a^n}{(ax + b)^{n+1}} .$$

(b) $f : x \rightarrow \sin x$

By the results of 2(a) and

$$\cos x = \sin(x + \tfrac{\pi}{2}) ,$$

$$-\sin x = \sin(x + \pi) ,$$

$$-\cos x = \sin(x + \tfrac{3\pi}{2}) ,$$

we have

$$D^n \sin x = \sin(x + \tfrac{n\pi}{2}) .$$

(c) $f : x \rightarrow \cos(ax + b)$

$$f'(x) = -a \sin(ax + b) = a \cos(ax + b + \tfrac{\pi}{2}) .$$

$$f''(x) = -a^2 \sin(ax + b + \tfrac{\pi}{2}) = a^2 \cos(ax + b + \pi) .$$

In general,

$$f^{(n)}(x) = a^n \cos(ax + \tfrac{n\pi}{2}) .$$

(d) $f : x \rightarrow \sin^2 x$

$$f'(x) = 2 \sin x \cos x = \sin 2x .$$

Then,

$$f^{(n)}(x) = 2^{n-1} \sin\left(2x + \frac{(n - 1)\pi}{2}\right)$$

(see part b).

(e) $f : x \rightarrow \cos^3 2x$

By twice applying the addition formula for cosines, or, alternatively, from DeMoivre's Theorem (SMSG, Intermediate Mathematics, Chapter 12), we get

$$\cos 3x = 4 \cos^3 x - 3 \cos x$$

or

$$\cos 6x = 4 \cos^3 2x - 3 \cos 2x$$

or

$$\cos^3 2x = \frac{1}{4} \cos 6x + \frac{3}{4} \cos 2x .$$

Thus,

$$D^n \cos^3 2x = D^n\left(\frac{1}{4} \cos 6x + \frac{3}{4} \cos 2x\right)$$

$$= \frac{6^n}{4} \cos\left(6x + \frac{n\pi}{2}\right) + \frac{3}{4} \cdot 2^n \cos\left(2x + \frac{n\pi}{2}\right) .$$

(See part c.)

(f) $f : x \rightarrow x^n - [x]^n$

$$D^n(x^n - [x]^n) = D^n x^n - D^n [x]^n .$$

But $[x]^n$ is a constant function between its discontinuities; thus if x is not an integer,

$$D^n x^n - D^n [x]^n = D^n x^n = n!$$

If x is an integer, then f is discontinuous at x and so has no derivative.

(g) $f : x \rightarrow \dfrac{1}{x^2 - a^2}$

$$f(x) = \frac{1}{2a} \left(\frac{1}{x - a} - \frac{1}{x + a}\right) .$$

Then

$$D^n f(x) = \frac{(-1)^n n!}{2a} \left(\frac{1}{(x - a)^{n+1}} - \frac{1}{(x + a)^{n+1}}\right), \text{ by part (a).}$$

We could write this

$$f^{(n)}(x) = \frac{(-1)^n n! \left((x + a)^{n+1} - (x - a)^{n+1}\right)}{2a(x^2 - a^2)^{n+1}}$$

$$= \frac{(-1)^n n!}{(x^2 - a^2)^{n+1}} \left(\binom{n+1}{1}x^n + \binom{n+1}{3}x^{n-2}a^2 + \binom{n+1}{5}x^{n-4}a^4 + \ldots\right).$$

(h) $f : x \to \dfrac{1}{x^2 + a^2}$

We use the result of (g) and replace a^2 by $-a^2$. The result may be verified by mathematical induction.

4. Let f be a function defined for all values of x with the property,

$$f(a + b) = f(a) \cdot f(b), \qquad \text{for all } a \text{ and } b.$$

(a) Show that either $f(0) = 1$ or $f(x) = 0$ for all x.

$$f(x) = f(x + 0) = f(x)f(0),$$

whence

$$f(x) - f(x)f(0) = 0,$$

or

$$f(x)\Big(1 - f(0)\Big) = 0.$$

So either $f(0) = 1$ or $f(x) = 0$.

(b) If $f(0) = 1$, show that $f(x) \neq 0$ for all x.

$$f(0) = 1$$

$$f(0) = f(x - x) = f(x)f(-x).$$

Thus, if $f(x) = 0$ then $f(0) = 0$; contradiction.

(c) If, additionally, $f(0) = 1$ and f has a derivative at $x = 0$, show that f' exists for all x and that $f'(x) = f'(0) \cdot f(x)$.

First we observe that

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(h)}{h}$$

$$= \lim_{h \to 0} \frac{f(h) - 1}{h}.$$

Also,

$$\frac{f(x+h)-f(x)}{h} = \frac{f(x)\cdot f(h)-f(x)}{h}$$

$$= f(x)\cdot\frac{f(h)-1}{h}\ ,$$

and

$$\lim_{h\sim 0} f(x)\cdot\frac{f(h)-1}{h} = \lim_{h\sim 0} f(x)\ \lim_{h\sim 0}\frac{f(h)-1}{h}$$

$$= f(x)\cdot f'(0)\ .$$

Consequently, for all x ,

$$\lim_{h\sim 0}\frac{f(x+h)-f(x)}{h} = f'(x) = f(x)\cdot f'(0)\ .$$

5. If $F(x+y) = F(x) + F(y)$, where F is continuous at one point and defined for all x , show that $F(x)$ is everywhere differentiable.

The proof proceeds from three lemmas:

(1) Continuity of F at any point, say x_0 , implies continuity of F at every point.

(2) $F(r\cdot x) = r\cdot F(x)$, r any rational $\frac{p}{q}$, x any real number.

(3) Lemmas 1 and 2 show that $F(kx) = kF(x)$, k and x any real numbers.

This condition (Lemma 3) is called homogeneity. Lemma 3 (homogeneity) is sufficient to show differentiability at all points; for suppose F , as given above, is homogeneous. Then

$$F(x+h) - F(x) = F(x) + F(h) - F(x)$$

$$= F(h)$$

$$= hF(1)\ .$$

Then for all x ,

$$\lim_{h\sim 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\sim 0}\frac{h\cdot F(1)}{h}$$

$$= F(1)\ .$$

So F' exists for all x .

Now we will prove the three lemmas.

Lemma 1. Continuity at x_0 implies continuity for all x.

Proof. Given $\epsilon > 0$, we have by continuity at x_0 that for $|h| < \delta$ $|F(x_0 + h) - F(x_0)| < \epsilon$. But $F(x_0 + h) - F(x_0) = F(h) = F(x + h) - F(x)$, for any x. So F is continuous at any point x.

Lemma 2. $F(rx) = rF(x)$, r rational and x any real number.

Proof.

$$\begin{aligned} F(2x) &= F(x + x) \\ &= F(x) + F(x) \\ &= 2F(x) . \end{aligned}$$

Similarly, $F(nx) = nF(x)$ for n any natural number. This is easily shown by mathematical induction (Section A3-1). Now set $y = \frac{x}{n}$. Then

$$F(ny) = nF(y)$$

or

$$F(x) = nF(\tfrac{x}{n})$$

or

$$\frac{1}{n} F(x) = F(\tfrac{x}{n}) , \qquad n \text{ any natural number.}$$

Combining these two facts in one equation yields

$$F(\tfrac{p}{q} \cdot x) = \frac{p}{q} F(x) , \quad p \text{ and } q \text{ any natural numbers}$$

or

$$F(rx) = rF(x) \qquad r \text{ rational.}$$

Lemma 3. $F(kx) = kF(x)$ for all real numbers x and k.

Proof. First we will show that $F(h) = hF(1)$ for any real number h. Let $F(h) = hF(1) + a$. We will show $a = 0$. Assume $a \neq 0$. Then

$$\begin{aligned} |F(r) - F(h)| &= |r \cdot F(1) - h \cdot F(1) - a| \qquad \text{by Lemma 2} \\ &= |a - (r - h)F(1)| \\ &\geq |a| - |(r - h)F(1)| \\ &\geq |a| - |r - h| \cdot |F(1)| . \end{aligned}$$

So for any $\epsilon < |a|$, if $|r - h| < \frac{|a| - \epsilon}{|F(1)| + 1}$ then $|a| - |r - h| \cdot |F(1)| > \epsilon$, and hence $|F(r) - F(h)| > \epsilon$. But by continuity of F at h, $|F(r) - F(h)| < \epsilon$ if $|r - h|$ is small enough; contradiction.

So $a = 0$ and $F(h) = hF(1) + 0 = h \cdot F(1)$ for any h. But this is sufficient to show homogeneity:

$$\begin{aligned} F(kx) &= kx \cdot F(1) \\ &= k \cdot F(x) . \end{aligned}$$

So F is homogeneous and, consequently, differentiable..

6. Given that the function F is defined for all x, is differentiable in $[0,a)$, and satisfies $F(x + a) = kF(x)$ for all x (a, k constants). Show that F is everywhere differentiable and sketch some possible graphs of F assuming
   (a) $F(0) = 0$, $F'(0) = 0$.
   (b) $F(0) = 0$, $F'(0) = 1$.
   (c) If $F(0) = 0$ and the graph of F is tangent to the y-axis, show that F is differentiable for all $x \neq 0$ and sketch some possible graphs of F.

If x is any real number, then for some integer t and some real number x^* such that $0 \le x^* < a$,

$$x = ta + x^* .$$

Then,

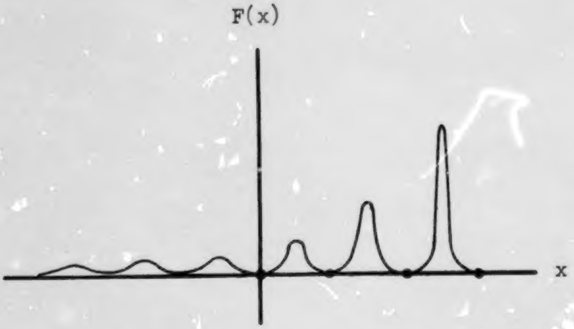
$$\begin{aligned} \frac{F(x + h) - F(x)}{h} &= \frac{F(ta + x^* + h) - F(ta + x^*)}{h} \\ &= \frac{k^tF(x^* + h) - k^tF(x^*)}{h} \\ &= k^t \cdot \frac{F(x^* + h) - F(x^*)}{h} . \end{aligned}$$

Since F is differentiable at x^*, we get, on letting $h \to 0$, that F is differentiable everywhere and

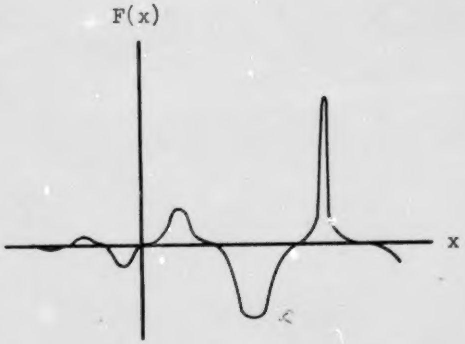
$$F'(x) = k^tF'(x^*) .$$

This proof does not work for the function described in part (c) because there F is not differentiable at every point in $[0,a)$, but only at every point in $(0,a)$.

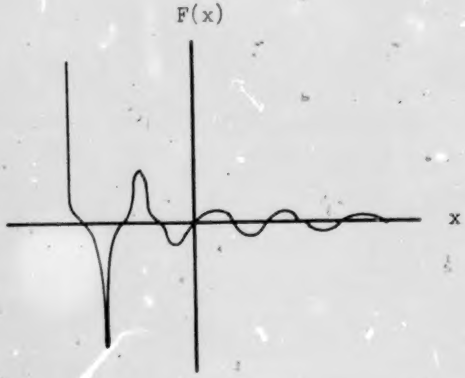
Note that if $k = 1$, F is a periodic function.



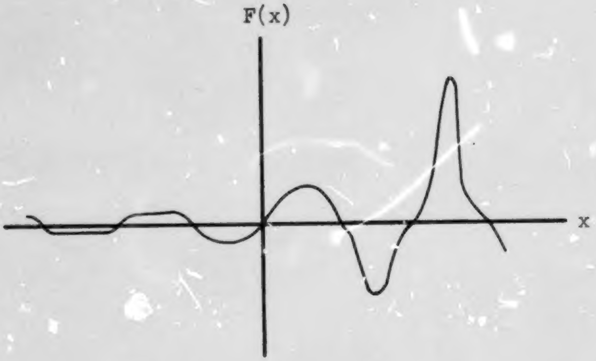
Condition (a), $K > 2$, $K \approx 2$.



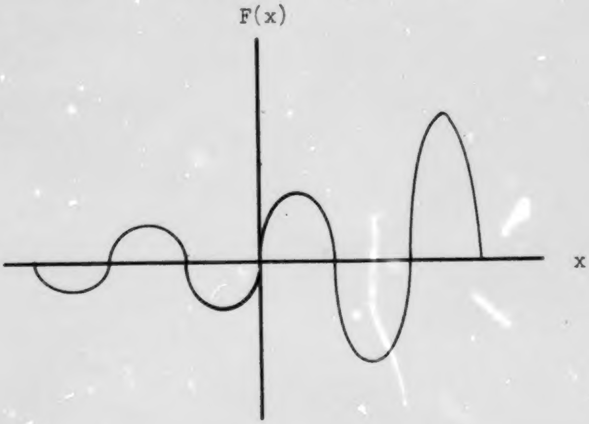
Condition (a), $K < 0$, $K \approx -2$.



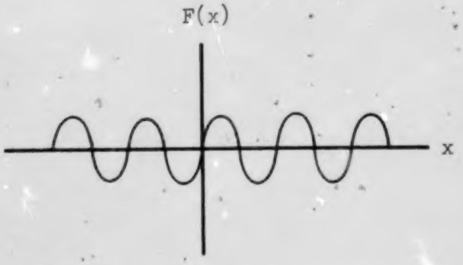
Condition (b), $0 < K < 1$, $K \approx \frac{1}{2}$.



Condition (b), $K < 0$, $K \approx -2$.



Condition (c), $K < 0$, $K \approx -\frac{3}{2}$.



Condition (c), $K > 0$, $K \approx 1$.

TC5-2. The Derivative at an Extremum.

We distinguish between an extremum (a value of the function) and the point (or value x) at which the function takes on an extremum.

Theorem 5-2a. It is as important to know what Theorem 5-2a does not imply as it is to know what it does imply. It does not imply that f has an extremum; it does not imply that f has an extremum at u.

Theorem 5-2b. The statement of the theorem is geometrically plausible. On the other hand, the geometric interpretation does not illuminate the proof but may actually cause confusion since the method of proof is indirect.

Corollaries 1 and 2 to Theorem 5-2b provide a convenient tool for checking possible extrema (under certain conditions). The two corollaries share four common hypotheses: the continuity and differentiability of f; existence of u with $f'(u) = a$; $f(u) \neq f(a)$, $f(u) \neq f(b)$. In each corollary the remaining hypothesis compares the values of f at u with the values at the endpoints, a, b.

Sometimes simplification in locating extrema can be achieved by substitutions, by discarding unnecessary constants, and by squaring. Thus, for example, the value of x which maximizes $\frac{f(x)}{c}$, c constant, $c \neq 0$, also maximizes $f(x)$. (In Example 5-6a we say, "... we may equivalently minimize $f(x,y)$"; justification for this simplification and many others is established in Exercises 5-5, No. 4.)

Solutions Exercises 5-2a

1. Complete the proof of Theorem 5-2a by proving that $f'(u) \not< 0$.

Suppose $f'(u) < 0$. Then since

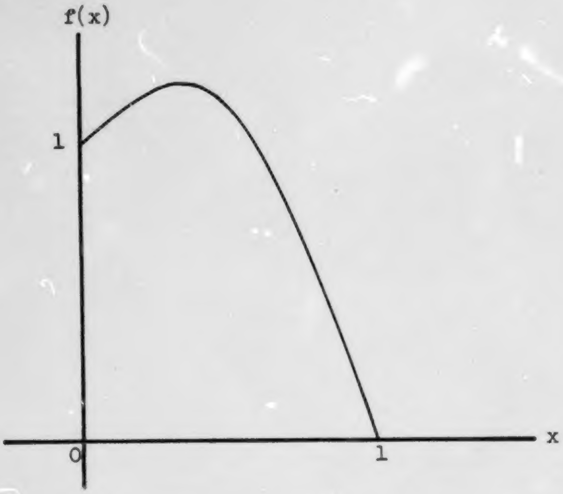
$$f'(u) = \lim_{x \to u} \frac{f(x) - f(u)}{x - u},$$

there is a neighborhood I of u, by Lemma 3-4, such that

$$r(x) = \frac{f(x) - f(u)}{x - u} < 0, \qquad \text{for } x \in I.$$

Since u is an interior point of $[a,b]$, there will be points α and β in the interval I for which $\alpha < u < \beta$ and such that $r(\alpha) < 0$ and $r(\beta) < 0$. Consequently, $f(\alpha) > f(u) > f(\beta)$, and therefore $f(u)$ cannot be an extremum.

2. Make a careful sketch on the interval $[0,1]$ of the graph of the function $f : x \longrightarrow 1 + x - x^2 - x^3$ given in Example 5-2a. Does the graph confirm the conclusions of the text?

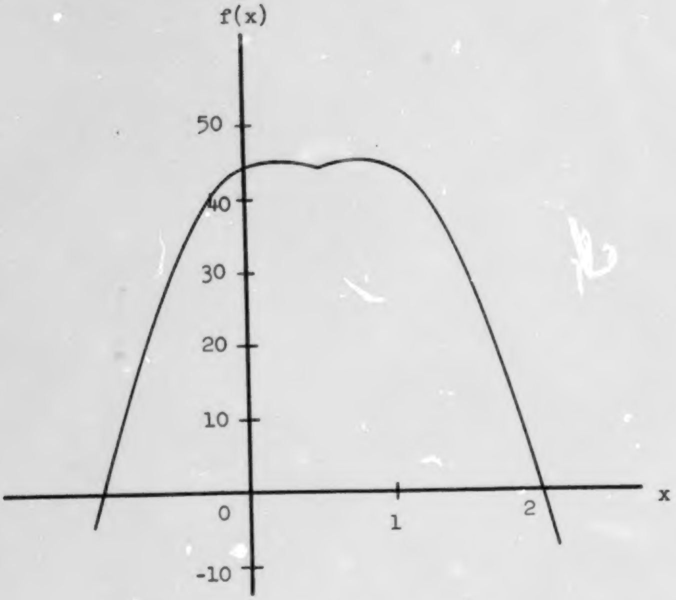


$$f : x \longrightarrow 1 + x - x^2 - x^3$$

Yes, the graph verifies the conclusions of the text.

3. Determine the extreme values of the function $f : x \rightarrow 44 + 4x - 13x^2 + 18x^3 - 9x^4$ and make a careful sketch of the graph of f. Compare your results with your answer to Exercises 1-1, Number 12.

$f'(x) = 4 - 26x + 54x^2 - 36x^3 = (x - \frac{1}{2})(x - \frac{1}{3})(x - \frac{2}{3})$. Thus, the maxima or minima are at $x = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$. $f(\frac{1}{3}) = f(\frac{2}{3}) = 44\frac{4}{9}$, $f(\frac{1}{2}) = 44\frac{7}{16}$ but $f(1) = 44$ and $f(-1) = 0$. Therefore, $f(\frac{1}{3})$ and $f(\frac{2}{3})$ are global maxima, and $f(\frac{1}{2})$ is a local minimum.



$$f(x) = 44 + 4x - 13x^2 + 18x^3 - 9x^4$$

4. Locate and characterize the extreme values of each of the following functions on the interval $[-1,1]$.

   (a) $f : x \longrightarrow x^{2/3}$

   Maximum values at the endpoints $x = -1$ and $x = 1$. We have $f(-1) = 1$ and $f(1) = 1$. Minimum value at the interior point $x = 0$; here $f(x) = 0$ but $f'(x) = \frac{2}{3}x^{-1/3}$ is not defined.

   (b) $f : x \longrightarrow |x|^{3/2}$

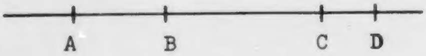
   Maximum values at the endpoints $x = -1$ and $x = 1$; $f(-1) = 1$ and $f(1) = 1$. Minimum value at $x = 0$, $f(0) = 0$. Note also that $f'(0) = 0$. (See TC Exercises 2-2, No. 3b, 3c.)

5. (a) Three men live on the same straight road. Where on the road should they agree to meet so that the sum of the distances they travel along the road from their homes to their meeting place is to be a minimum?

We suppose that no two men live in the same house. Given any two men, the sum of the distances they travel to meet at any given point between them is independent of the point. Thus the point should be shown to minimize the travel at the third man, i.e., at the home of the man who lives between the other men.

(b) What is the answer if the number of men is four?

If the four men live at points A , B , C , D , respectively, they can meet at points B or C or at any point on the road between B and C .



(c) Answer the question for n men where n is any positive integer.

If n is odd, they should meet at the house which is the $(\frac{n}{2} + \frac{1}{2})$th from either end. If n is even, they can meet at the house at $\frac{n}{2}$th or $(\frac{n}{2} + 1)$th place from either end or at any point on the road in between these houses. If we allow more than one man to live in the same house then we order the distances down the road for the n men as follows:

$$x_1 \leq x_2 \leq x_3 \cdots \leq x_n$$

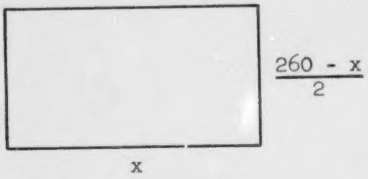
where x_k denotes the distance of the house of the k-th man with respect to some origin on the road. If n is odd, the answer is as before, at x_k where $k = \frac{(n + 1)}{2}$. If n is even, and if it happens that $x_j = x_{j+1}$ where $j = \frac{n}{2}$, then the answer is $x_j = x_{j+1}$. If $x_j \neq x_{j+1}$, then as before the answer is at x_j , x_{j+1} or any point in between. (Also see Exercises A1-3, Nos. 5d and 5c.)

6. A stone wall 100 yards long stands on a ranch. Part or all of it is to be used in forming a rectangular corral, using an additional 260 yards of fencing for the other three sides. Find the maximum area which can be so enclosed.

The area of the corral is given by

$$A = f(x) = 130\,x - \frac{x^2}{2} \ ,$$

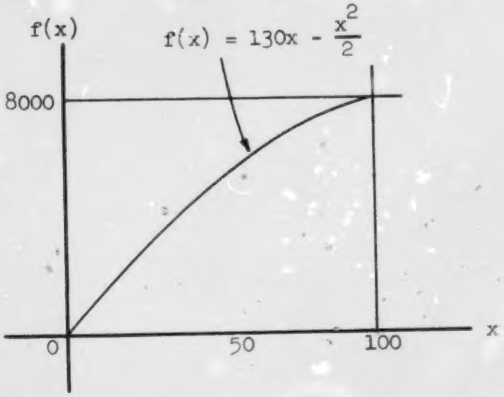
where $x \le 100$ and $x \ge 0$.



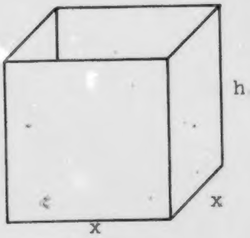
$$f'(x) = 130 - x \quad \text{is zero if} \quad x = 130 \ .$$

$$f(0) = 0 \ , \ f(100) = 8000 \ .$$

Since $x \le 100$, the extrema occur at the endpoints and the maximum area occurs at $x = 100$. The maximum area is 8000 sq. yds.



7. A metal can with square base and open top is required to contain a gallon (231 cubic inches) of gasoline. Neglecting the thickness of the metal and the waste material in construction, find the dimensions that require the least material.



$V = x^2h = 231$ so that $h = \frac{231}{x^2}$.

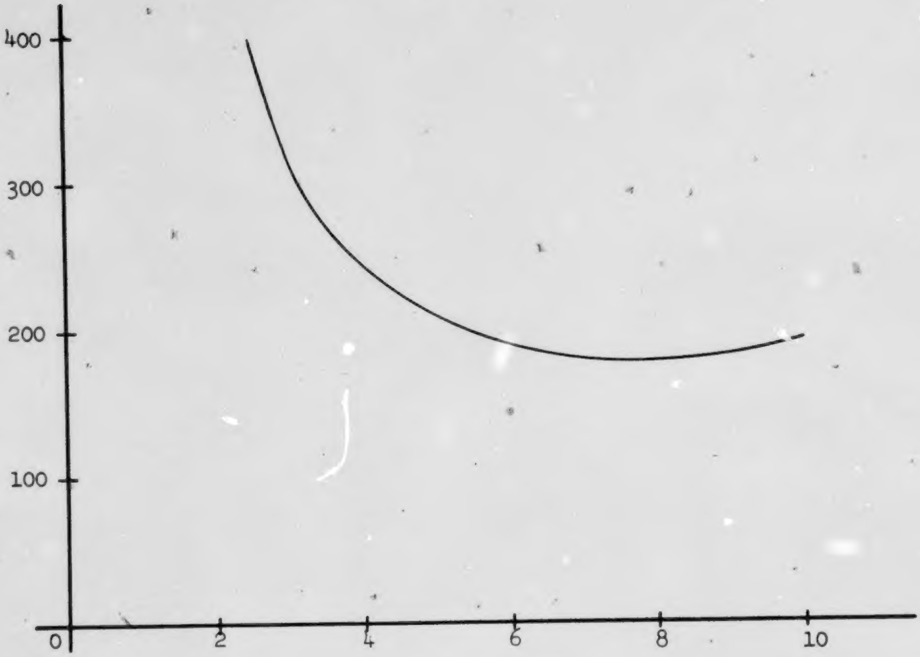
The surface area, S , is given by

$S = x^2 + 4xh$, so

$$S = f(x) = x^2 + \frac{924}{x} \ , \ x > 0 \ .$$

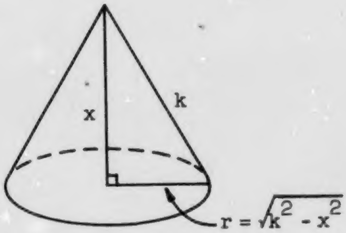
$f'(x) = 2x - \frac{924}{x^2} = 0$ if and only if $x = \sqrt[3]{462}$. The function f has only one extremum at $x = \sqrt[3]{462}$. For $x \sim 0^+$ (Exercises 3-4, No. 16) and x sufficiently large, f(x) can be made arbitrarily large. Thus, the extremum is a minimum. (See graph of f.) The dimensions that require the minimum material are

$$\sqrt[3]{462} \text{ by } \sqrt[3]{462} \text{ by } \frac{1}{2}\sqrt[3]{462} \qquad \text{(in inches)}.$$



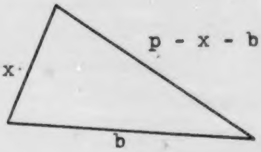
$$S = f(x) = x^2 + \frac{924}{x}$$

8. A right triangle with hypotenuse k is rotated about one of its legs. Find the maximum volume of the right circular cone produced.



The volume of the cone is given by
$V = \frac{\pi x}{3}(k^2 - x^2) = f(x)$, where
$0 < x < k$. $f'(x) = \frac{\pi}{3}(k^2 - 3x^2) = 0$
if $x = \frac{k}{\sqrt{3}}$.
$f\left(\frac{k}{\sqrt{3}}\right) = \frac{2\sqrt{3}\,\pi k^3}{27}$, $f(0) = 0$, and
$f(k) = 0$. Thus, the maximum volume is at the interior point $x = \frac{k}{\sqrt{3}}$
The maximum volume is $\frac{2\sqrt{3}\,\pi k^3}{27}$.

9. Determine the lengths of the sides of a triangle of maximum area with base b and perimeter p. (Hint: use Heron's formula for the area of a triangle: $A = \sqrt{s(s - a)(s - b)(s - c)}$, where a, b, c are the lengths of the sides, and $s = \frac{1}{2}(a + b + c) = \frac{p}{2}$.)



$$A = f(x) = \sqrt{s(s - x)(x - b)(x + b - s)}\ ,\ x \le s$$

$$f'(x) = \frac{(x + b - s)(s)(b - s) + s(s - b)(s - x)}{2\sqrt{s(s - x)(s - b)(x + b - s)}}$$

$$= \frac{s(b - s)(2x + b - 2s)}{2\sqrt{s(s - x)(s - b)(x + b - s)}}$$

$f'(x) = 0$ if $b = s = \frac{p}{2}$ or if $x = \frac{p - b}{2}$. If $x = \frac{p}{2}$, the area is zero. The maximum area is given by an isosceles triangle with two equal sides $\frac{p - b}{2}$.

Solutions Exercises 5-2b

1. Prove the corollaries to Theorem 5-2b.

Corollary 1. Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). If there exists only one point u in (a,b) where $f'(u) = 0$ and if either $f(a) < f(u) < f(b)$ or $f(a) > f(u) > f(b)$ then $f(u)$ is not a local extremum.

Consider the case where $f(a) < f(u) < f(b)$ (proof for $f(a) > f(u) > f(b)$ is similar). Suppose $f(u)$ is a local extremum on (a,b), then by Theorem 5-2b f is increasing on one of the closed intervals $[a,u]$ and $[u,b]$ and decreasing on the other. Therefore, either $f(a) \le f(u)$ and $f(u) \ge f(b)$, or $f(a) \ge f(u)$ and $f(u) \le f(b)$. But this is a contradiction.

Corollary 2. Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). Let there be only one point u in the open interval where $f'(u) = 0$. If $f(u) > f(a)$ and $f(u) > f(b)$ then $f(u)$ is the maximum of f on $[a,b]$. If $f(u) < f(a)$ and $f(u) < f(b)$ then $f(u)$ is the minimum of f on $[a,b]$.

$f'(x) \neq 0$ on (a,u) and (u,b). Therefore (Theorem 5-2a), there are no extrema on these intervals. Then, by Theorem 5-2b, f is either increasing or decreasing on $[a,u]$ and $[u,b]$. If $f(u) > f(a)$ and $f(u) > f(b)$ then it follows that $f(u)$ is the maximum of f on $[u,b]$ and $f(u)$

is the maximum of f on $[a,u]$. Therefore, $f(u)$ is the maximum of f on $[a,b]$.

Similarly, if $f(u) < f(a)$ and $f(u) < f(b)$, then $f(u)$ is the minimum of f on $[a,b]$.

2. For each of the following functions locate and characterize all extrema. On what intervals is the function increasing? decreasing?

(a) $f : x \longrightarrow 4x^4 - 8x^2 + 1$

$f(x) = 4x^4 - 8x^2 + 1$.
$f'(x) = 16x^3 - 16x = 16x(x^2 - 1)$ is zero if $x = 0$ or $x = 1$ or $x = -1$. Since $f(-1) = f(1) = -3$ and $f(0) = 1$, by Lemma 5-2 we conclude that $f(0)$ is a local extremum; $f(0) = 1$ is a local maximum. Similarly, since $f(-2) = f(2) = 33$ we may conclude that $f(-1) = f(1) = -3$ are minima. The function f is decreasing throughout the intervals $x \leq -1$ and $[0,1]$; f is increasing throughout the intervals $[-1,0]$ and $x \geq 1$.

(b) $f : x \longrightarrow x^4 - 4x^3$

$f(x) = x^4 - 4x^3$.
$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$ is zero if $x = 0$ or $x = 3$. $f(0) = 0$ and $f(3) = -27$. Since $f(-1) > f(0) > f(3)$, $f(0)$ is not a local extremum, by the Corollary to Theorem 5-2b. Since $f(0) = f(4)$, $f(3)$ is an extremum by Lemma 5-2. $f(3)$ is a minimum value of f. The function is decreasing throughout the interval $x \leq 3$ and increasing throughout the interval $x \geq 3$.

(c) $f : x \longrightarrow \dfrac{x^3}{1 + x^2}$

$f(x) = \dfrac{x^3}{1 + x^2}$.

$f'(x) = \dfrac{x^4 + 3x^2}{(1 + x^2)^2} = 0$ if and only if $x = 0$. $f(0) = 0$, $f(-1) = -\frac{1}{2}$, $f(1) = \frac{1}{2}$. Thus, by Corollary to Theorem 5-2b, $f(0)$ is not an extremum. The function f is increasing for all real numbers, as $f'(x) > 0$.

(d) $f : x \longrightarrow \dfrac{x}{x^2 - 1}$

$f(x) = \dfrac{x}{x^2 - 1}$.

$f'(x) = \dfrac{-(x^2 + 1)}{(x^2 - 1)^2}$. Since $f'(x) = 0$ has no solution, f has no extremum, but is decreasing throughout the intervals $x < -1$, $(-1,1)$, and $x > 1$. ($f'(x) < 0$ for all x , $|x| \neq 1$).

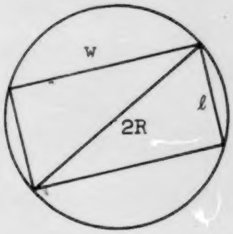
(e) $f : x \longrightarrow \dfrac{x}{1 + x^2}$

$f(x) = \dfrac{x}{1 + x^2}$.

$f'(x) = \dfrac{1 - x^2}{(1 + x^2)^2}$ is zero if $x = 1$ or $x = -1$. $f(-1) = -\frac{1}{2}$ is a minimum value and $f(1) = \frac{1}{2}$ is a maximum value of f . The function f is increasing throughout the closed interval $[-1,1]$ and decreasing throughout the intervals $x \leq -1$ and $x \geq 1$.

3. A rectangle is inscribed in a circle of radius R . Find the rectangle of maximum area. of maximum perimeter.

From the rectangle problem, Exercises 1-1, Number 2, we have $A = f(w) = w\sqrt{4R^2 - w^2}$ on the interval $[0,2R]$, i.e., $0 \leq w \leq 2R$, where w is the length of one side of the rectangle. The derivative $f'(w) = \dfrac{4R^2 - 2w^2}{\sqrt{4R^2 - w^2}}$ is zero at $w = \sqrt{2}\,R$. We note that $f(0) = f(2R) = 0$ and $f(\sqrt{2}\,R) = 2R^2$ and conclude that the maximum area is taken on at the interior point $\sqrt{2}\,R$. Thus, the rectangle of maximum area is a square of side $\sqrt{2}\,R$.



$\ell = \sqrt{4R^2 - w^2}$

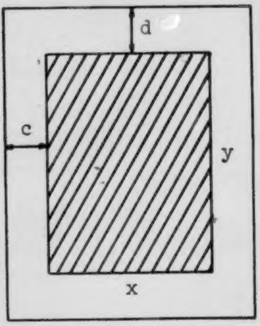
For the maximum perimeter,

$$P = 2(\ell + w)$$
$$= 2(\sqrt{4R^2 - w^2} + w) = g(w) .$$

The derivative $g'(w) = \dfrac{-2w}{\sqrt{4R^2 + w^2}} + 2$ is zero at $w = \sqrt{2}\,R$. Since $g(0) = 4\sqrt{2}\,R$, we conclude that the maximum perimeter is taken on at the interior point $\sqrt{2}\,R$. The rectangle of maximum perimeter is a square of side $\sqrt{2}\,R$.

4. The area of the printed text on a page is A square centimeters. The left and right margins are each c centimeters wide, and the upper and lower margins are each d centimeters. What are the most economical dimensions of the pages if only the amount of paper matters?

Let the total area of the page be T.



$A = xy$

$$T = (x + 2c)(y + 2d)$$

$$= (x + 2c)\left(\frac{A}{x} + 2d\right) = f(x)$$

$$f'(x) = \frac{2dx^2 - 2Ac}{x^2}$$

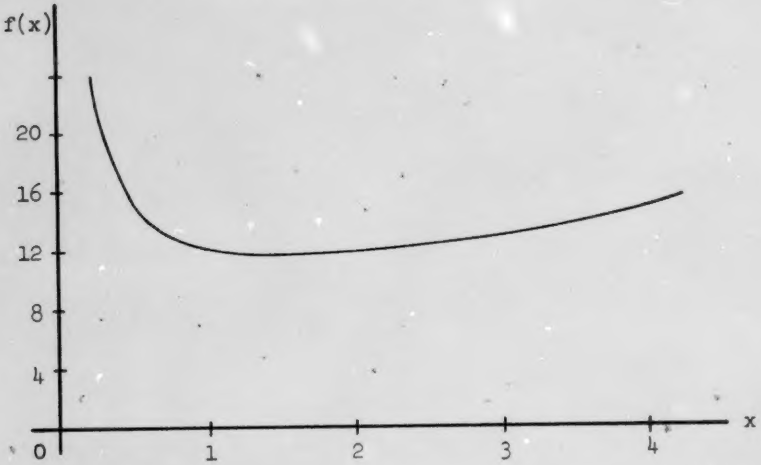
is zero at $x = \sqrt{\frac{cA}{d}}$.

$$f'(x) > 0 \text{ if } x^2 > \frac{cA}{d}, \text{ and}$$

$$f'(x) < 0 \text{ if } x^2 < \frac{cA}{d}.$$

So $x = \sqrt{\frac{cA}{d}}$ is a minimum, and the most economical dimensions of the pages are

$$\sqrt{\frac{cA}{d}} + 2c \text{ by } \sqrt{\frac{dA}{c}} + 2d, \qquad \text{in centimeters.}$$



$T = f(x) = (x + 2c)\left(\frac{A}{x} + 2d\right)$, where $c = d = 1$, $A = 2$.

5. A rectangle has two of its vertices on the x-axis and the other two above the axis on the parabola $y = 6 - x^2$. What are the dimensions of such a rectangle if its area is to be a maximum?

(See Exercises 1-1, No. 6.)

$$A = f(x) = 12x - 2x^3 \quad , \quad 0 \leq x \leq \sqrt{6}$$

$$f'(x) = 12 - 6x^2 = 0 \quad \text{if} \quad x = \sqrt{2}$$

$$f(0) = f(\sqrt{6}) = 0 \quad \text{and} \quad f(\sqrt{2}) = 8\sqrt{2}$$

The area is a maximum if the dimensions are $2\sqrt{2}$ by 4.

6. A rectangular sheet of galvanized metal is bent to form the sides and bottom of a trough so that the cross section has this shape: ⊔ . If the metal is 14 inches wide, how deep must the trough be to carry the most water?

(See Exercises 1-1, No. 7.)

$$V = f(d) = 14d - 2d^2 \quad , \quad 0 \leq d \leq 7 \; ,$$

where d is the depth.

$$f'(d) = 14 - 4d = 0 \quad \text{if} \quad d = \frac{7}{2} \; ,$$

$$f(0) = f(7) = 0 \quad \text{and} \quad f\left(\frac{7}{2}\right) = \frac{49}{2} \; .$$

The trough should be $3\frac{1}{2}$ inches deep to carry the most water.

7. Find the right circular cylinder of greatest volume that can be inscribed in a right circular cone of radius r and height h.

(See Exercises 1-1, No. 9.)

$$v = f(x) = \pi h x^2 - \frac{\pi h x^3}{r} \quad , \quad 0 \leq x \leq r \; ,$$

where x is the radius of the cylinder.

$$f'(x) = 2\pi h x - \frac{3\pi h x^2}{r} = 0 \quad \text{if} \quad x = 0 \quad \text{or} \quad x = \frac{2}{3}r \; .$$

$$f(0) = f(r) = 0 \quad \text{and} \quad f\left(\frac{2}{3}r\right) = \frac{4}{27}\pi r^2 h \; .$$

The right circular cylinder of greatest volume has a radius which is $\frac{2}{3}$ of the radius of the cone and a height which is $\frac{1}{3}$ of the height of the cone.

8. The lower right-hand corner of a page is folded over so as to reach the left edge in such a way that one endpoint of the crease is on the right-hand edge of the page and the other endpoint is on the bottom edge of the page, as in the figure. If the width of the page is c inches, find the minimum length of the crease.

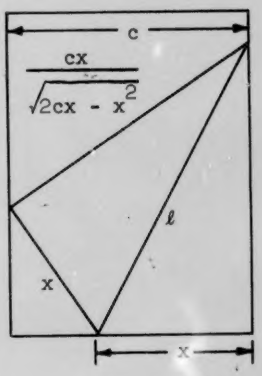
(See Exercises 1-1, No. 11.)

We note, using similar triangles and the Pythagorean Theorem, that

$\ell^2 = f(x) = \frac{2x^3}{2x - c}$, for $x > \frac{c}{2}$.

$f'(x) = \frac{8x^3 - 6x^2c}{(2x - c)^2}$ is zero if

$x = 0$ or $x = \frac{3}{4}c$. Since $x = 0$ is not in the domain of f we consider only $x = \frac{3}{4}c$ for extremum.



$$f\left(\frac{3}{4}c\right) = \frac{27}{16}c^2 .$$

$f(.6c) = 2.16c^2$ and $f(c) = 2c^2$.

We conclude that $f\left(\frac{3}{4}c\right)$ is a minimum value. The minimum length of the crease is $\frac{3\sqrt{3}}{4}c$.

9. What is the smallest positive value of t such that the slope of $y = 2\sin\left(\frac{t}{2} - \frac{\pi}{3}\right)$ is zero?

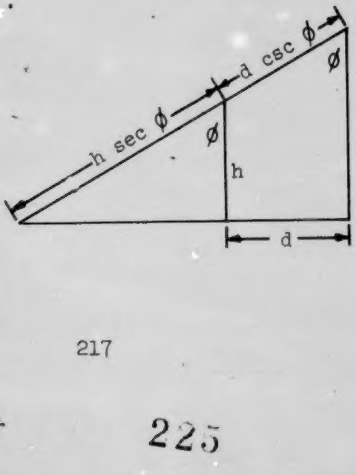
$y = f(t) = 2\sin\left(\frac{t}{2} - \frac{\pi}{3}\right)$.

$f'(t) = \cos\left(\frac{t}{2} - \frac{\pi}{3}\right)$ is zero if $\frac{t}{2} - \frac{\pi}{3} = \frac{\pi}{2}$, i.e., for $t = \frac{5\pi}{3}$.

10. A wall h feet high stands d feet away from a tall building. A ladder L feet long reaches from the ground outside the wall to the building. Let ϕ be the angle between the ladder and the building.

(a) Show that if the ladder touches the top of the wall, $L = d \csc \phi + h \sec \phi$.

Refer to the figure.



(b) Find the shortest ladder that will reach the building if $h = 8$ and $d = 24$.

$f(\phi) = 8 \sec \phi + 24 \csc \phi$, if $h = 8$ and $d = 24$.

$f'(\phi) = 8(\sec \phi \tan \phi - 3 \csc \phi \cot \phi)$ is zero if $\tan^3 \phi = 3$, i.e., $\phi = \arctan 3^{1/3}$. For this value of ϕ, $f(\phi) = 8(3^{2/3} + 1)^{3/2}$. The shortest ladder has length $8(3^{2/3} + 1)^{3/2} \approx 44$. To see that f is a minimum at $\phi = \arctan 3^{1/3}$ we find $f(\arctan \frac{1}{2}) = 28\sqrt{5}$ and $f(\arctan 4) = 14\sqrt{17}$, for example. Clearly, each of these values is greater than $f(\arctan 3^{1/3})$. Thus, by Corollary 2 to Theorem 5-2b, $f(3^{1/3})$ is a minimum value.

11. In an experiment repeated n times, one obtains the numbers $a_1, a_2, \ldots, a_n$ for a certain physical quantity x. What value of x should we take if we want to:

(a) minimize the sum of the squares of the deviations, i.e.,

$$(x - a_1)^2 + (x - a_2)^2 + \ldots + (x - a_n)^2 \ ?$$

Let $f(x) = (x - a_1)^2 + (x - a_2)^2 + \ldots + (x - a_n)^2$. Then $f'(x) = 2(x - a_1) + 2(x - a_2) + \ldots + 2(x - a_n)$. $f'(x) = 0$ when $x = \dfrac{a_1 + a_2 + \ldots + a_n}{n}$ (the arithmetic mean of the a_i's). Since the graph of f is a parabola

$$\left(f(x) = nx^2 - 2x \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} a_i^2 \right), \text{ and } n > 0,$$

$f(x)$ is minimum when x is the arithmetic mean.

(b) What value of x should we take if we want to minimize the sum of the absolute values of the deviations, i.e.,

$$|x - a_1| + |x - a_2| + \ldots + |x - a_n| \ ?$$

Take $f(x) = |x - a_1| + |x - a_2| + \ldots + |x - a_n|$. Note that f is continuous everywhere and differentiable except when $x = a_r$, $r = 1, 2, \ldots, n$. We have

$$f'(x) = \sum_{i=1}^{n} \operatorname{sgn}(x - a_i), \qquad (x \neq a_i).$$

Since

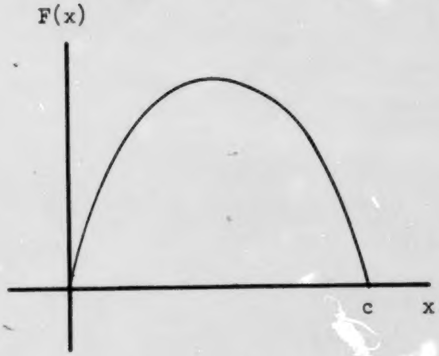
$$\operatorname{sgn}(x = a_i) = \begin{cases} 1, & x > a_i \\ -1, & x < a_i \end{cases}$$

we see that $f'(x)$ cannot be zero unless there are as many minuses as pluses; i.e., the number of terms is even, $n = 2m$, and $a_m < x < a_{m+1}$. If $x < a_m$ we see that the number of pluses is less than the number of minuses, $f'(x) < 0$, and f is decreasing in each interval $[a_i , a_{i+1}]$ where $i < m$ and $a_i \neq a_{i+1}$. Thus f is decreasing whenever $x < a_m$. Similarly, when $x > a_m$, the number of pluses exceeds the number of minuses and f is increasing. Since f is constant for $a_m \leq x \leq a_{m+1}$ it follows that all such values of x yield the global minimum.

Similarly, if n is odd, $n = 2m - 1$, we conclude that f is decreasing for $x < a_m$ and increasing for $x > a_m$. It follows that the minimum occurs at $x = a_m$.

12. Find the maximum of $x^m y^m$ (m, n rational and greater than 0) if $x + y = c$ (c constant) and $x \geq 0$, $y \geq 0$.

Note that this statement includes results established in Exercises 5-2a, Numbers 8 and 9, and Exercises 5-2b, Numbers 3, 5, 6, 7, and 11, as special cases.



$$x^m y^n = x^m (c - x)^n .$$

Let $F(x) = x^m (c - x)^n$; then

$$F'(x) = mx^{m-1}(c - x)^n - nx^m (c - x)^{n-1} .$$

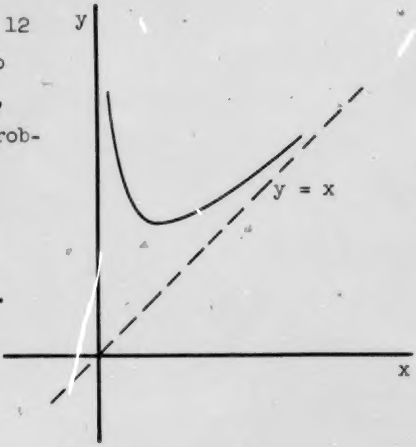
If $F'(x) = 0$, then

$$x^{m-1}(c - x)^{m-1}\Big(m(c - x) - nx\Big) = 0 .$$

$x = 0$ and $x = c$ are endpoints whose function values are zero. The maximum is achieved where $m(c - x) - nx = 0$, or where $x = \frac{mc}{m + n}$. Thus $y = \frac{mc}{m + n}$, and

the maximum is $$m^m n^n \left(\frac{c}{m + n}\right)^{m+n} .$$

13. Find the minimum of $x + y$ if $x^m y^n = k$ (k constant, m and n rational and greater than 0).



The maximizing ratio $\frac{x}{y}$ in Number 12 is the same as the minimizing ratio $\frac{x}{y}$ in Number 13. (Exercises 5-2a, No. 7 is a special case of this problem.)

$$x + y = x + \frac{k^{1/n}}{x^{m/n}} = F(x) .$$

$$F'(x) = 1 - \frac{m}{n} k^{1/n} x^{-(m/n)-1} .$$

If $F'(x) = 0$, then

$$x = \left(\frac{m}{n} k^{1/n}\right)^{n/(m+n)}$$

$$y = \frac{k^{1/n}}{\left(\frac{m}{n} k^{1/n}\right)^{m/(m+n)}} .$$

TC5-3. The Law of the Mean.

Rolle's Theorem (Lemma 5-3) is named for Michel Rolle (1652-1719), a French contemporary of Newton and Leibnitz. In some versions of the Theorem the end values, $f(p)$ and $f(q)$, are taken to be zero.

The Law of the Mean (Theorem 5-3) is a generalization of Rolle's Theorem: it does not assume that $f(p) = f(q)$.

In many other texts the Law of the Mean is referred to as the Mean Value Theorem of the Differential Calculus as opposed to the Mean Value Theorem of the Integral Calculus which we call simply the Mean Value Theorem. (We do not prove the Mean Value Theorem in the text, but leave it to the exercises since it is not used here to prove any later theorem.)

The Law of the Mean would be false if there were theoretical gaps in the set of real numbers. The proof of the theorem depends on the completeness of the real number system. More specifically, the proof may be traced, in turn, through Lemmas 5-3, 5-2, Theorem 5-2a, Lemma 3-4, Theorems A4-1, 3-7a, b, and the Nested Interval Principle (A1-5) to the Separation Axiom.

Geometric intuition suggests that the Law of the Mean should follow immediately from Rolle's Theorem. The inherent difficulties are discussed in the text. However, the intuition remains valid; compare with the Generalized Law of the Mean (Chapter 11). Consider a differentiable arc (Chapter 11) given in parametric form by the equations

$$x = F(t) \quad , \quad y = G(t) \quad , \qquad \text{for } t_1 \le t \le t_2 \ ,$$

where F and G are continuous on the closed interval $[t_1,t_2]$ and differentiable on the open interval (t_1,t_2). If $F'(t)$ and $G'(t)$ are not simultaneously zero in (t_1,t_2), then there is a value u in (t_1,t_2) for which the slope of the curve is the same as the slope of the chord joining the endpoints, $\big(F(t_1), G(t_1)\big)$ and $\big(F(t_2), G(t_2)\big)$. If (say) $F'(u) \neq 0$, this statement takes the form

$$\frac{G'(u)}{F'(u)} = \frac{G(t_2) - G(t_1)}{F(t_2) - F(t_1)} \quad \text{(Generalized Law of the Mean).}$$

Statement of the Generalized Law of the Mean: Let F and G be functions continuous on $[a,b]$ and differentiable on (a,b). There exists a value u in (a,b) where

$$G'(u)[F(b) - F(a)] = F'(u)[G(b) - G(a)] \ .$$

The Law of the Mean is the special case with $G(x) = x$.

There are functions that do not satisfy the hypothesis of Theorem 5-3 for which the conclusion of the Theorem holds.

The Law of the Mean is of utmost importance since it is used in the proof of most of the theorems of differential calculus. The theorem should be thoroughly understood and permanently remembered by each student of calculus.

Example TC5-1 shows that $f'(a)$ may exist although $\lim_{x \to a} f'(x)$ does not exist. We now prove that if $\lim_{x \to a} f'(x)$ exists, then $f'(a)$ exists. By our hypothesis, $f'(x)$ exists for all x in a deleted neighborhood of a.

Let $L = \lim_{x \to a} f'(x)$. Given $\epsilon > 0$, there is a δ so that

$$|f'(x) - L| < \epsilon \quad \text{if} \quad 0 < |x - a| < \delta$$

Let $r(x) = \dfrac{f(x) - f(a)}{x - a}$ if $0 < |x - a| < \delta$. By the Law of the Mean

$$r(x) = f'(\xi) \quad , \quad 0 < |\xi - a| < |x - a| < \delta \ ,$$

hence

$$|r(x) - L| = |f'(\xi) - L| < \epsilon \quad \text{if} \quad 0 < |x - a| < \delta$$

and

$$f'(a) = L .$$

Solutions Exercises 5-3

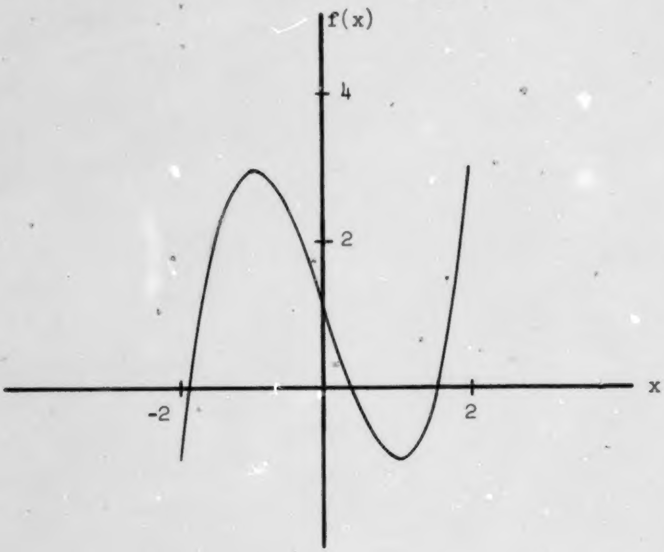
1. Prove Corollary 2 to Lemma 5-3.

   Corollary 2. A polynomial of degree n can have no more than n distinct real roots.

   Let $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$, $a_n \neq 0$. The n-th derivative of p is given by $p^{(n)}(x) = n!a_n$, a nonzero constant function. It follows by Corollary 1 to Lemma 5-3 that $p^{(n-1)}$ has at most one real root. Applying this argument recursively we obtain the desired result.

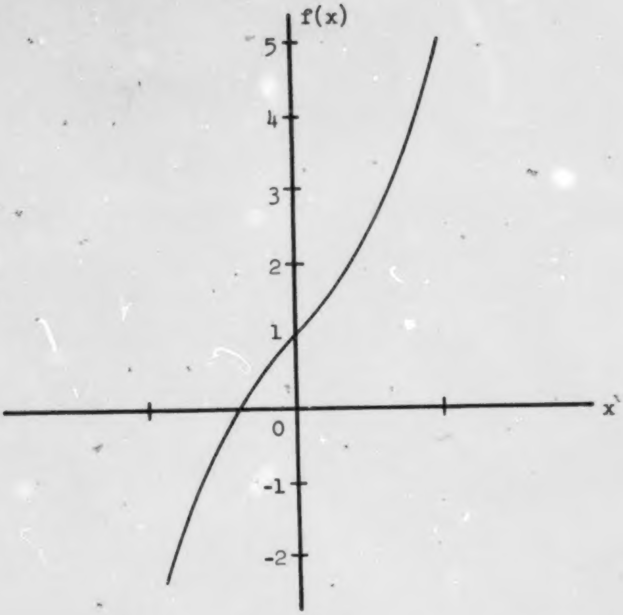
2. Sketch the graphs of the functions in Example 5-3a.

(i)



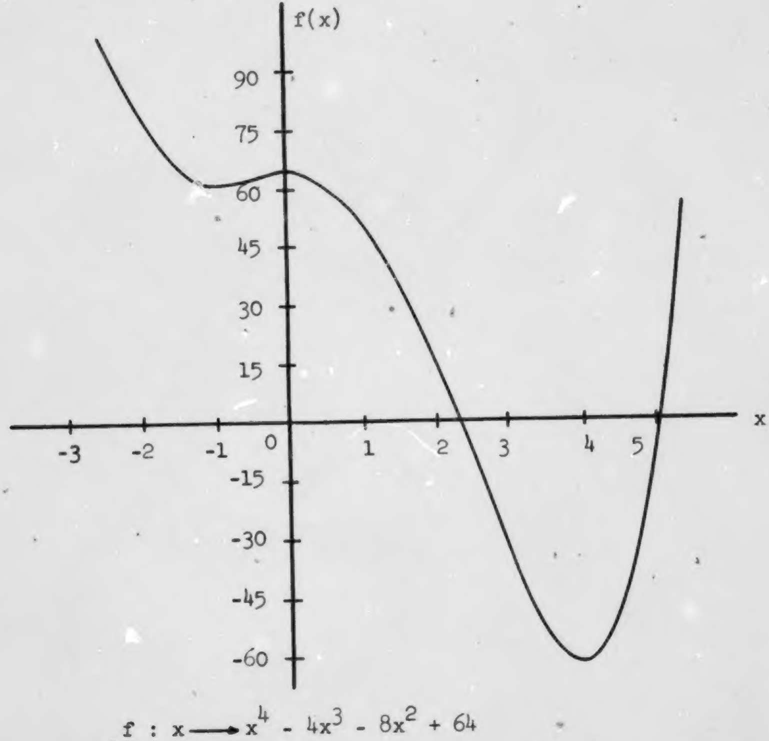
$f : x \longrightarrow x^3 - 3x + 1$

(ii)



f : $x \longrightarrow x^3 + 3x + 1$

(iii)



f : $x \longrightarrow x^4 - 4x^3 - 8x^2 + 64$

3. Is the following converse of Rolle's Theorem true? If f is continuous on the closed interval $[p,q]$ and differentiable on the open interval (p,q), and if there is at least one point u in the open interval where $f'(u) = 0$, then there are two points m and n where $p \leq m < u < n \leq q$ such that $f(m) = f(n)$.

Not true. Counterexample: $y = x^3$ for any interval containing $x = 0$ in its interior.

4. Does Rolle's Theorem justify the conclusion that $\frac{dy}{dx} = 0$ for some values of x in the interval $-1 \leq x \leq 1$ for $(y + 1)^3 = x^2$?

If $(y + 1)^3 = x^2$, $\frac{dy}{dx} = \frac{2x}{3(y + 1)^2} = \frac{2}{3x^{1/3}}$. The conclusion of Rolle's Theorem does not hold for the closed interval $[-1,1]$ since $\frac{dy}{dx}$ does not exist at $x = 0$.

5. Given: $f(x) = x(x - 1)(x - 2)(x - 3)(x - 4)$. Determine how many solutions $f'(x) = 0$ has and find intervals including each of these without calculating $f'(x)$.

By Corollary 1 to Lemma 5-3, $f'(x) = 0$ has four solutions. There is one solution in each of the open intervals $(0,1)$, $(1,2)$, $(2,3)$, and $(3,4)$ since the zeros of f are $0, 1, 2, 3, 4$.

6. Verify that Rolle's Theorem (Lemma 5-3) holds for the given function in the given interval or give a reason why it does not.

(a) $f : x \longrightarrow x^3 + 4x^2 - 7x - 10$, $[-1,2]$

(b) $f : x \longrightarrow \frac{2 - x^2}{x^4}$, $[-1,1]$

(a)

$$f(x) = x^3 + 4x^2 - 7x - 10$$

$$f'(x) = 3x^2 + 8x - 7$$

$$f(-1) = f(2) = 0$$

One of the zeros of f', $\frac{-4 + \sqrt{37}}{3}$, is in the interval $(-1,2)$. Thus, Rolle's Theorem holds.

(b) $f : x \longrightarrow \frac{2 - x^2}{x^4}$

The function is not continuous at $x = 0$ and hence the theorem does not apply.

7. Prove that the equation

$$f(x) = x^n + px + q = 0$$

cannot have more than two real solutions for an even integer n nor more than three real solutions for an odd n. Use Rolle's Theorem. This problem can also be done without it (Exercises 3-7, No. 21).

$$f(x) = x^n + px + q$$

$$f'(x) = nx^{n-1} + p = 0$$

If n is even, f' is of odd degree and $f'(x) = 0$ has one and only one solution. It follows from Corollary 1 to Lemma 5-3 that f cannot have more than two real solutions in this case. Similarly, if n is odd, f' is of even degree and $f'(x) = 0$ has no more than two solutions. In this case $f(x) = 0$ has no more than three real solutions.

8. A function g has a continuous second derivative on the closed interval $[a,b]$. The equation $g(x) = 0$ has three different solutions in the open interval (a,b). Show that the equation $g''(x) = 0$ has at least one solution in the open interval (a,b).

By Corollary 1 to Lemma 5-3, g' has at least two zeros in the open interval (a,b) and hence the derivative of g', which is g'', must have at least one zero in the interval (a,b).

9. Show that the conclusion of the Law of the Mean does not follow for $f(x) = \tan x$ in the interval $1.5 < x < 1.6$ and explain why.

The theorem does not apply. The function f is not continuous on the open interval $(1.5, 1.6)$. (Note that $1.5 < \frac{\pi}{2} < 1.6$ and $f(x)$ is not defined at $\frac{\pi}{2}$.) $\tan(1.5) > 0$, $\tan(1.6) < 0$. Therefore, if the Law of the Mean held in the interval $[1.5, 1.6]$, there would exist a u in $(1.5, 1.6)$ such that $f'(u) < 0$. But $D \tan x = \sec^2 x$, which is nonnegative.

10. For each of the following functions show that the Law of the Mean fails to hold on the interval $[-a,a]$ if $a > 0$. Explain why the theorem fails.

(a) $f : x \longrightarrow |x|$

$f(-a) = f(a) = a$. Yet $f'(x)$ is either $+1$ or -1 and never zero, so the Law of the Mean fails to hold. ($f'(0)$ does not exist.)

(b) $f : x \longrightarrow \frac{1}{x}$

If the Law of the Mean holds for f on the interval $[-a,a]$, then for some u in $(-a,a)$, $f'(u) = \frac{4}{a^2}$. But $f'(x) = -\frac{1}{x^2}$ and so is never positive. (f is not continuous at $x = 0$.)

11. Show that the equation $x^5 + x^3 - x - 2 = 0$ has exactly one solution in the open interval $(1,2)$.

$f(x) = x^5 + x^3 - x - 2$ and $f'(x) = 5x^4 + 3x^2 - 1$. Since $f(1) < 0$ and $f(2) > 0$ the function f has a zero in the interval $(1,2)$, by the Intermediate Value Theorem and since $f'(x) > 0$ for $x > 1$, the function f has only one zero in the interval.

12. Show that $x^2 = x \sin x + \cos x$ for exactly two real values of x.

Let $f(x) = x^2 - x \sin x - \cos x$. Then

$$f'(x) = 2x - x \cos x$$
$$= x(2 - \cos x).$$

$f'(x) = 0$ if and only if $x = 0$ and hence f has no more than two zeros. Since $f(0) = -1$ and $f(\pi) = f(-\pi) = \pi^2 + 1$ we conclude that there are zeros in the open intervals $(-\pi,0)$ and $(0,\pi)$, by the Intermediate Value Theorem.

13. Find a number that can be chosen as the number u in the Law of the Mean for the given function and interval.

(a) $f : x \longrightarrow \cos x$, $0 \le x \le \frac{\pi}{2}$

$u = \arcsin(\frac{2}{\pi})$

(b) $f : x \longrightarrow x^3 \ , \ -1 \le x \le 1$

$$u = -\frac{\sqrt{3}}{3} \text{ or } \frac{\sqrt{3}}{3}$$

(c) $f : x \longrightarrow x^3 - 2x^2 + 1 \ , \ -1 \le x \le 0$

$$u = \frac{2 - \sqrt{13}}{3}$$

(d) $f : x \longrightarrow \cos x + \sin x \ , \ 0 \le x \le 2\pi$

$$u = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

14. Derive each of the following inequalities by applying the Law of the Mean.

(a) $|\sin x - \sin y| \le |x - y|$

If $f(x) = \sin x$, then f is continuous and differentiable for all x. By the Law of the Mean, for any $x \neq y$,

$$\frac{\sin x - \sin y}{x - y} = \cos u \ , \quad \text{where} \quad x < u < y \ ,$$

and hence

$$\left|\frac{\sin x - \sin y}{x - y}\right| = |\cos u| \le 1 \ ,$$

or

$$|\sin x - \sin y| \le |x - y| \ .$$

(b) $\dfrac{x}{1 + x^2} < \arctan x < x \quad \text{if} \quad x > 0$.

If $g(x) = \arctan x$, g is continuous for $x \ge 0$ and differentiable for all $x > 0$. By the Law of the Mean,

$$\frac{\arctan x - \arctan 0}{x} = \frac{1}{1 + u^2} \ , \quad \text{where} \quad 0 < u < x \ .$$

Since

$$\frac{1}{1 + x^2} < \frac{1}{1 + u^2} < 1 \ ,$$

we have

$$\frac{1}{1 + x^2} < \frac{\arctan x}{x} < 1 \ ,$$

so that

$$\frac{x}{1 + x^2} < \arctan x < x \quad , \quad x > 0 \ .$$

15. Use the Law of the Mean to approximate $\sqrt[3]{1.008}$.

Here $f(x) = \sqrt[3]{x}$ and we can choose $p = 1$, $q = 8$ for numerical simplicity. If we approximate $f(x)$ by the linear function g whose graph is the chord joining the points $(1,1)$ and $(8,2)$,

$$g(x) = 1 + \frac{1}{7}(x - 1) .$$

Thus, an approximate value of $\sqrt[3]{1.008}$ is given by $g(1.008) = 1.001$.

Since $|g(x) - f(x)| \leq 2M_1(x - p)$ where M_1 is an upper bound of $f'(x)$ in $[p,q]$, we have the error extimate

$$|1.001 - \sqrt[3]{1.008}| \leq 2(.008) \cdot \max \frac{1}{3x^{2/3}}$$

or

$$|1.001 - \sqrt[3]{1.008}| < 2(.008)(\tfrac{1}{3}) < .0054$$

and

$$0.9957 < \sqrt[3]{1.008} < 1.0064 .$$

To get a better approximation, we choose a number q closer to 1 . A convenient value is $q = \frac{27}{8}$. Then

$$g(x) = 1 + \frac{\frac{3}{2} - 1}{\frac{27}{8} - 1}(x - 1) = 1 + \frac{4}{19}(x - 1) ,$$

and

$$g(1.008) = 1 + \frac{4}{19}(.008) \approx 1.0017 .$$

Here

$$|1.0017 - \sqrt[3]{1.008}| \leq 2(.008)\frac{1}{3}$$

and

$$.9963 < \sqrt[3]{1.008} < 1.0071 .$$

Combining this with the previous approximation, we have

$$.9963 < \sqrt[3]{1.008} < 1.0064 .$$

We can sharpen the upper bound by choosing q very close to 1 . get arbitrarily close to $\sqrt[3]{1.008}$.

Alternatively, we can proceed as follows. From the Law of the Mean,

$$\frac{f(q) - f(p)}{q - p} = f'(u) , \qquad p < u < q ,$$

it follows that

$$\frac{\sqrt[3]{1.008} - \sqrt[3]{1}}{1.008 - 1} = \frac{1}{3\sqrt[3]{u^2}} , \qquad 1 < u < 1.008 ,$$

(here $f(x) = \sqrt[3]{x}$, $q = 1.008$, $p = 1$). Then, since $f'(u)$ is monotonic in $[1, 1.008]$, we have

$$\frac{.008}{3\sqrt[3]{\frac{27}{8}}} < \frac{.008}{3\sqrt[3]{1.008^2}} < \sqrt[3]{1.008} - 1 < \frac{.008}{3} ,$$

or

$$1.0017 < \sqrt[3]{1.008} < 1.0027 .$$

16. Use the Law of the Mean to approximate $\cos 61^\circ$.

$$\frac{\cos q - \cos p}{q - p} = -\sin u , \qquad p < u < q .$$

Choose

$$p = \frac{\pi}{3} \quad , \quad q = \frac{\pi}{3} + \frac{\pi}{180} .$$

Note that p and q are in radians and $1^\circ = \frac{\pi}{180}$ radians.

$$\frac{\cos(\frac{\pi}{3} + \frac{\pi}{180}) - \cos\frac{\pi}{3}}{\frac{\pi}{180}} = -\sin u .$$

Since

$$\sin 60^\circ < \sin u < 1 ,$$

$$-\frac{\pi}{180} \cdot 1 < \cos 61^\circ - \frac{1}{2} < -\frac{\pi}{180} \cdot \frac{1}{2}$$

or

$$\frac{1}{2} - \frac{\pi}{180} < \cos 61^\circ < \frac{1}{2} - \frac{\pi}{360} .$$

17. Show that $a\left(1 + \frac{\epsilon}{n(a^n + \epsilon)}\right) < \sqrt[n]{a^n + \epsilon} < a(1 + \frac{\epsilon}{na^n})$ for $\epsilon > 0$, $a > 1$, $n > 1$ (n rational) .

Let $f(x) = \sqrt[n]{x}$. Then

$$\frac{f(q) - f(p)}{q - p} = f'(u) , \qquad p < u < q .$$

Choose $q = a^n + \epsilon$, $p = a^n$. Thus,

$$\frac{\sqrt[n]{a^n + \epsilon} - \sqrt[n]{a^n}}{\epsilon} = \frac{u^{1/n}}{n \cdot u} , \qquad a^n < u < a^n + \epsilon ,$$

and if $a > 1$,

$$\frac{\epsilon \sqrt[n]{a^n + \epsilon}}{n(a^n + \epsilon)} < \sqrt{a^n + \epsilon} - a < \frac{\epsilon a}{na^n} .$$

Since

$$\frac{\epsilon \sqrt[n]{a^n + \epsilon}}{n(a^n + \epsilon)} > \frac{\epsilon a}{n(a^n + \epsilon)} ,$$

we obtain the desired inequalities.

18. Using Number 17, obtain the following approximations:

(a) $3 + \frac{1}{10} < \sqrt[3]{30} < 3 + \frac{1}{9}$.

If $n = a = \epsilon = 3$ in the result of Number 17, then

$$3 + \frac{1}{10} < \sqrt[3]{30} < 3 + \frac{1}{9} .$$

(b) $3 + \frac{3}{5(244)} < \sqrt[5]{244} < 3 + \frac{1}{405}$.

If we set $n = 5$, $a = 3$, $\epsilon = 1$ in the result of Number 17, we obtain the desired result.

(c) Show that the approximation

$$\frac{1}{2} \left(3 + \frac{3}{5(244)} + 3 + \frac{1}{405}\right) \text{ to } \sqrt[5]{244}$$

is correct to at least five decimal places.

We note that, in general, if $a < b < c$, then the error in taking $\frac{a + c}{2}$ as an approximation to b is no greater than $\frac{1}{2} (c - a)$. Letting $a = 3 + \frac{3}{5(244)}$, $b = \sqrt[5]{244}$, and $c = 3 + \frac{1}{405}$,

$$\frac{1}{2} (\frac{1}{405} - \frac{3}{5(244)} = \frac{(1220 - 1215)}{2(1215)(1220)} = \frac{1}{2(81)(1220)} < \frac{1}{100,000} .$$

Therefore,

$$|\frac{1}{2} (3 + \frac{1}{405} + 3 + \frac{3}{5(244)} - \sqrt[5]{244}| < 10^{-5} .$$

∧19. (a) Show that a straight line can intersect the graph of a polynomial function of n-th degree at most n times.

Let $y = mx + b$ be an equation of a straight line and $p(x) = a_0 + a_1x + \dots + a_nx^n$ be an n-th degree polynomial $(a_n \neq 0)$. Then if $y = p(x)$, we have $g(x) = 0$, where $g(x) = (a_0 - b) + (a_1 - m)x + a_2x^2 + \dots + a_nx^n$, $a_n \neq 0$. By Corollary 2 to Lemma 5-3 (proved in Solution No. 1), $g(x)$, a polynomial of n-th degree, has at most n distinct real roots. Therefore, a straight line can intersect the graph of a polynomial function of n-th degree at most n times (unless, of course, $g(x)$ is identically 0 ; i.e., p is linear and $p(x) = mx + b$).

(b) Obtain the corresponding result for rational functions.

Let $R(x) = \frac{f(x)}{g(x)}$ where f is of degree $t \geq 0$ and g of degree $s \geq 0$, $g(x) \neq 0$. When $R(x) = p(x)$, where p is a polynomial function of degree n, then $p(x) \cdot g(x) - f(x) = 0$. Let $q(x) = p(x) \cdot g(x) - f(x)$. Then the degree of q is at most $\max\{t, s + n\}$. Thus the graph of a rational function $x \longrightarrow R(x) = \frac{f(x)}{g(x)}$, where f is of degree $t \geq 0$ and g is of degree $s \geq 0$, can intersect the graph of a polynomial of degree n in at most $\max\{t, s + n\}$ points. Since $n = 1$ for the linear function p, we conclude that the graph of a straight line can meet the graph of R in at most $\max\{t, s + 1\}$ points. (Again, if R is linear there is an exceptional case.)

(c) Could $\sin x$ or $\cos x$ be rational functions? Justify your answer.

No. By the results of (b), if $\sin x$ were a rational function, then the equation $\sin x = 0$ would have a finite number of solutions. The same remark applies to $\cos x$.

∧20. Prove the intermediate value property for derivatives; namely, if f is differentiable on the closed interval $[p,q]$ then $f'(x)$ takes on every value between $f'(p)$ and $f'(q)$ in the open interval (p,q).

Suppose $f'(p) < m < f'(q)$. Set $\epsilon = \min\{m - f'(p), f'(q) - m\}$. There exists a value δ satisfying $0 < \delta < q - p$ for which, simultaneously,

$$\left|\frac{f(p + \delta) - f(p)}{\delta} - f'(p)\right| < \epsilon \quad \text{and} \quad \left|\frac{f(q) - f(q - \delta)}{\delta} - f'(q)\right| < \epsilon .$$

For the function

$$g(x) = \frac{f(x + \delta) - f(x)}{\delta} ,$$

where δ is fixed and satisfies the preceding conditions, it follows that

$$g(p) < m < g(q - \delta) .$$

The function $g(x)$ is continuous on the closed interval $[p , q - \delta]$ and therefore satisfies the intermediate value property on that interval. There must then exist a value r in $(p, q - \delta)$ such that

$$g(r) = \frac{f(r + \delta) - f(r)}{\delta} = m .$$

By the Law of the Mean we have for some value u, $f(r + \delta) - f(r) = \delta f'(u)$ where $r < u < r + \delta$. It follows that $f'(u) = m$.

Alternate Solution:

Let

$$r_p(x) = \frac{f(x) - f(p)}{x - p} \qquad \text{for } x \neq p ,$$

$$r_p(x) = f'(p) \qquad \text{for } x = p .$$

r_p is continuous on $[p,q]$ and, by the Intermediate Value Theorem, takes on all values between $f'(p)$ and $\frac{f(q) - f(p)}{q - p}$.

Similarly, let

$$r_q(x) = \frac{f(x) - f(q)}{x - q} \qquad \text{for } x \neq q ,$$

$$r_q(x) = f'(q) \qquad \text{for } x = q .$$

r_q is continuous and takes on all values between $\frac{f(q) - f(p)}{q - p}$ and $f'(q)$ on $[p,q]$. Now by the Law of the Mean, there exists a u_p such that $f'(u_p) = r_p(x)$ and a u_q such that $f'(u_q) = r_q(x)$ for all x in $[p,q]$. Since $r_p(x)$ and $r_q(x)$ between them take on all values between $f'(p)$ and $f'(q)$, it follows by the Law of the Mean that $f'(x)$ takes on all values between $f'(p)$ and $f'(q)$.

TC5-4. Applications of the Law of the Mean.

The Law of the Mean is useful in situations when we wish to derive a property of a function f from a property of the derivative f'.

Corollary 1 to Theorem 5-4a states the seemingly obvious proposition that if the direction of a curve is horizontal on an interval then the curve is a horizontal straight line. In kinematic terms, it states that an object which has zero velocity stays in the same place. (To the student this may seem like disappointingly small potatoes.) Perhaps it should be pointed out that it is worthwhile to check our analytical abstractions occasionally to see how closely they are related to the intuitive conceptions from which they have sprung. Further, we shall see in Chapter 7 that this seemingly insignificant corollary has important consequences.

Example 5-4a. How small is "sufficiently small?"

In order to guarantee that the factor $2\sqrt{2} - 20x^2 + 21x^5$ is positive we assure that

$$|20x^2 - 21x^5| < 2\sqrt{2} .$$

We suppose that $|x| < \delta$ and first impose the condition $\delta \leq 1$. It then follows that

$$\begin{aligned}|20x^2 - 21x^5| &\leq |20x^2| + |21x^5| \\ &\leq 20|x| + 21|x| \\ &< 41\,\delta .\end{aligned}$$

It is sufficient to take $\delta = \frac{2\sqrt{2}}{41}$.

Solutions Exercises 5-4

1. On what intervals is the function

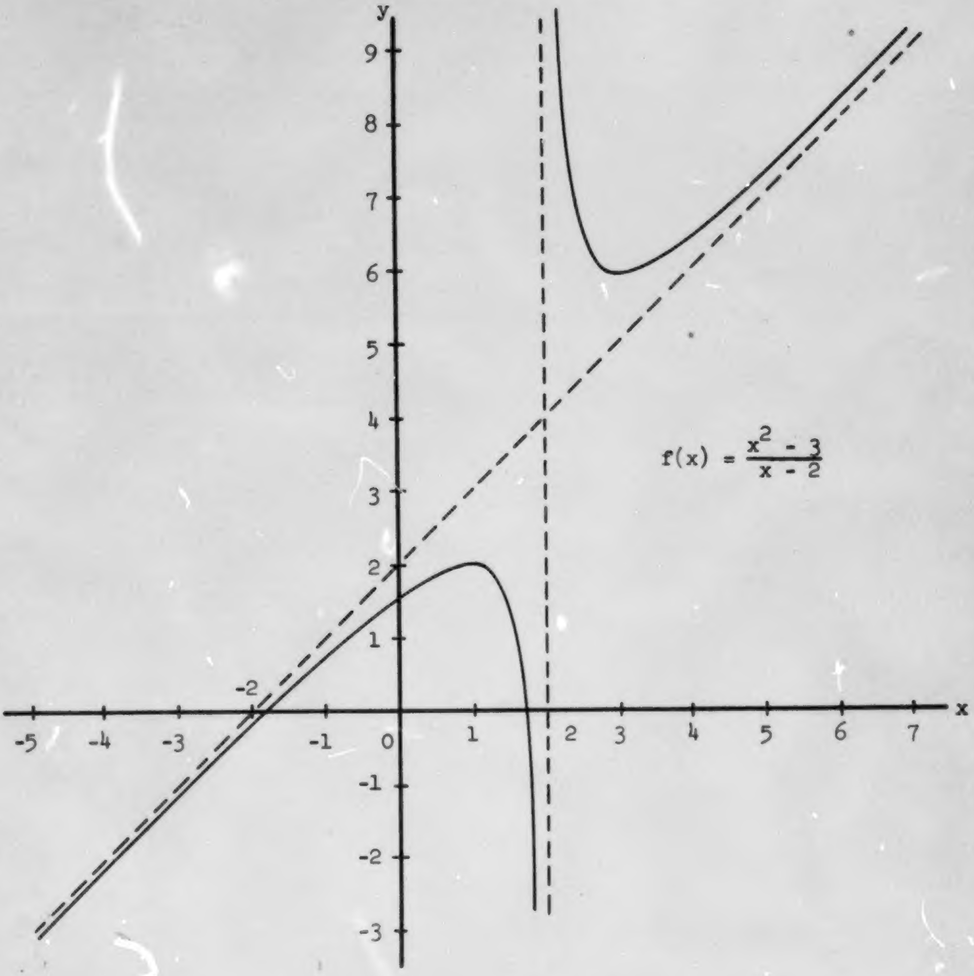
$$f : x \longrightarrow \frac{x^2 - 3}{x - 2}$$

strongly monotone? Use Theorem 5-4c to characterize all extrema.

$$f(x) = \frac{x^2 - 3}{x - 2} \quad \text{and} \quad f'(x) = \frac{(x - 1)(x - 3)}{(x - 2)^2}$$

By Theorem 5-4b, f is increasing for $x < 1$ and for $x > 3$, and f is decreasing for $1 < x < 2$ and for $2 < x < 3$.

By Theorem 5-4c, $f(1)$ is a maximum and $f(3)$ is a minimum.



2. Locate all intervals on which the function

$$f(x) = 44 + 4x - 13x^2 + 18x^3 - 9x^4$$

is increasing; decreasing. (Compare with your answer to Exercises 5-2a, No. 3.)

Refer to Exercises 1-1, Number 12, and TC Exercises 5-2a, Number 3.

$$f'(x) = 4 - 26x + 54x^2 - 36x^3$$

$$= -18(x - \tfrac{1}{3})(x - \tfrac{2}{3})(2x - 1)$$

f is increasing for $x < \frac{1}{3}$ and for $\frac{1}{2} < x < \frac{2}{3}$.

f is decreasing for $\frac{1}{3} < x < \frac{1}{2}$ and for $x > \frac{2}{3}$.

3. For each of the following functions find all points a for which $f'(a) = 0$. Examine the sign of f' and determine those intervals on which f is strongly monotone.

(a) $f(x) = \frac{x}{1 + x^4}$

$f(x) = \frac{x}{1 + x^4}$ and $f'(x) = \frac{1 - 3x^4}{(1 + x^4)^2}$.

$f'(x)$ is zero if $x = -\sqrt[4]{\frac{1}{3}}$, or $x = \sqrt[4]{\frac{1}{3}}$.

f is increasing if $|x| < \sqrt[4]{\frac{1}{3}}$ and decreasing if $|x| > \sqrt[4]{\frac{1}{3}}$.

(b) $f(x) = (1 - x)^4$

$f(x) = (1 - x)^4$ and $f'(x) = -4(1 - x)^3$.
$f'(1) = 0$. For $x > 1$, $f'(x) > 0$ and f is increasing; for $x < 1$, $f'(x) < 0$ and f is decreasing.

(c) $f(x) = (1 - x)^5$

$f(x) = (1 - x)^5$ and $f'(x) = -5(1 - x)^4$.
$f'(x) < 0$ for all $x \neq 1$ and hence by Theorem 5-4b f is decreasing on every interval not containing the point $x = 1$. (Actually, this is sufficient to show that f is decreasing on every interval; see comment following Theorem 5-4b, and Number 14 of this set of exercises.)

(d) $f(x) = \frac{x^2 + 10}{x^2 - 5}$

$f(x) = \frac{x^2 + 10}{x^2 - 5}$ and $f'(x) = \frac{-30x}{(x^2 - 5)^2}$.

$f'(0) = 0$; $f'(x) > 0$ for $x < 0$, where $x \neq -\sqrt{5}$, and hence f is increasing for $x < 0$ $(x \neq -\sqrt{5})$.
f is decreasing for $x > 0$ $(x \neq \sqrt{5})$.

4. If p and q are integers and

$$f(x) = (x - 1)^p(x + 1)^q \ , \ (p \geq 2 \, , q \geq 2)$$

find the extrema of f for the following cases:

(a) p and q are both even.

(b) p is even and q is odd.

(c) p is odd and q is even.

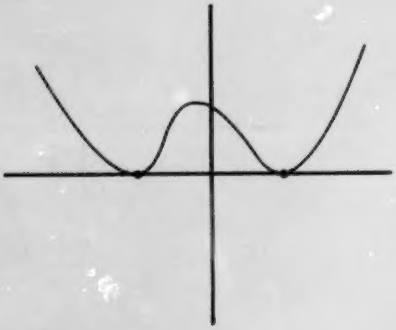
(d) p and q are both odd.

First observe on the basis of Lemma 5-2 that there is at least one extremum in the open interval $(-1,1)$. Next we note that the derivative vanishes at only one point in $(-1,1)$ so that this is the only extremum in the interval; namely,

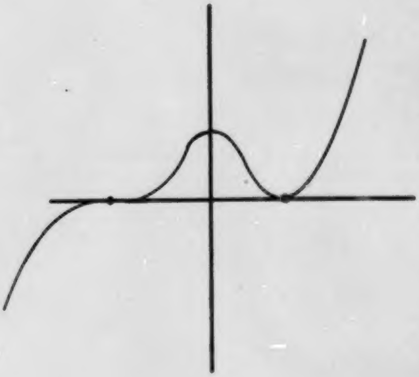
$$f'(x) = (x - 1)^{p-1}(x + 1)^{q-1}\Big((p + q)x - (q - p)\Big)$$

$$= \frac{1}{p + q}(x - 1)^{p-1}(x + 1)^{q-1}(x - \frac{q - p}{q - p}) \ ,$$

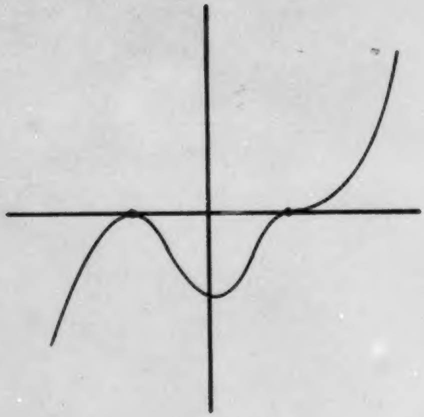
so that $f'(x) = 0$ in $(-1,1)$ only at $x = \frac{q - p}{q + p}$. Finally we observe that if q is even, the sign of the derivative changes at $x = 1$ and there is an extremum there; if q is odd, the sign does not change and there is an inflection. The same remark holds for p at $x = -1$. With this as a guide and knowing also how the sign of f changes at $x = \pm 1$ we can easily sketch the graphs in the four cases:



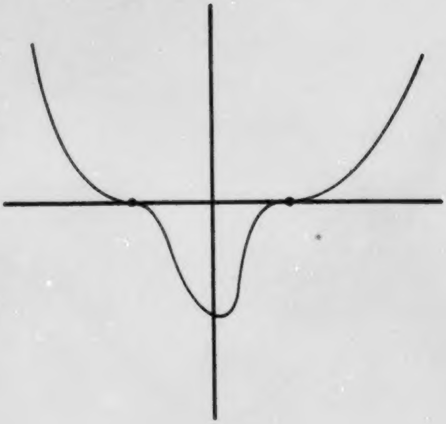
(a)



(b)



(c)



(d)

5. If p , q and r are positive integers, and $a < b < c$, discuss the graph of the function

$$f : x \longrightarrow (x - a)^p (x - b)^q (x - c)^r \, .$$

Discuss some special cases as in Number 4.

(a) Intercepts: $f(x) = 0$ at $x = a$, $x = b$, $x = c$.

$$f(0) = (-1)^{p+q+r} a^p b^q c^r \, .$$

(b) As x increases through large positive values, $f(x)$ grows large without bound; as x decreases through large negative values, $-f(x)$ grows large without bound if $p + q + r$ is odd and $f(x)$ grows large without bound if $p + q + r$ is even.

(c) By Lemma 5-2 we see that there is at least one extremum in each open interval (a,b) or (b,c) . Now

$$f'(x) = [p(x - b)(x - c) + q(x - a)(x - c) + r(x - a)(x - b)]$$
$$\cdot \, (x - a)^{p-1}(x - b)^{q-1}(x - c)^{r-1}$$
$$= Q(x)(x - a)^{p-1}(x - b)^{q-1}(x - c)^{r-1} \, .$$

Now $Q(x)$ is quadratic and can have no more than two roots; these roots must then correspond to the extrema in the intervals (a,b) and (b,c) .

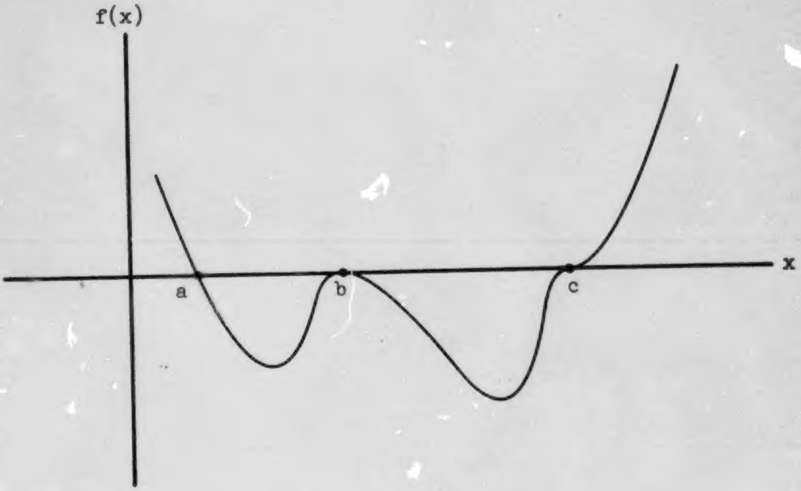
We can now easily analyze the behavior of f in terms of the behavior of $g(x)(x - \alpha)^n$ where $g(\alpha) \neq 0$ and $g(x)$ is differentiable. There are three basic cases to consider:

(i) n = 1 . The graph $y = g(x)(x - \alpha)$ crosses the x-axis at α without tangency.

(ii) n even. The graph $y = g(x)(x - \alpha)^n$ does not cross the x-axis at α , is tangent there and has constant sign in a deleted neighborhood of α . Hence $x = \alpha$ yields an extremum.

(iii) n odd, $n > 1$. The graph $y = g(x)(x - \alpha)^n$ crosses the x-axis at α and has a tangency there.

To sketch the graph of f we now observe the changes of sign in the function and use the foregoing. Since p , q , and r can satisfy the conditions (i), (ii), (iii) independently there are 27 distinct cases. We sketch the case $p = 1$, q even, and r odd, $r > 1$.



$f : x \longrightarrow (x - a)(x - b)^{2m}(x - c)^{2n+1}$

6. A tank is to have a given volume V and is to be made in the form of a right circular cylinder with hemispherical ends. The material for the ends costs twice as much per square foot as that for the cylindrical part. Find the most economical dimensions.

Refer to Exercises 1-1, Number 8.

The surface area of the tank, S , is given by

$$S = 4\pi r^2 + 2\pi r\ell$$
$$= \frac{4}{3}\pi r^2 + \frac{2V}{r},$$

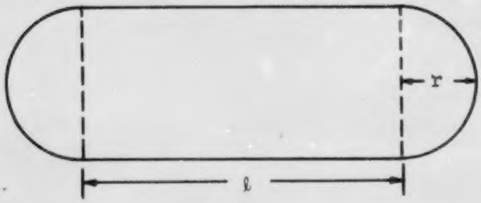
where V represents the volume.

If the cost of the ends is 2k per square foot, then the total cost, [illegible], is given by

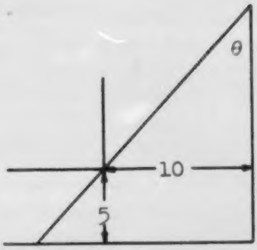
$$C = \frac{8}{3}k\pi r^2 + \frac{2kV}{r} = f(r).$$

$$f'(r) = 2k\left(\frac{8}{3}\pi r - \frac{V}{r^2}\right) = 2k\left(\frac{8\pi r^3 - 3V}{3r^2}\right)$$

is zero if $r = \sqrt[3]{\frac{3V}{8\pi}}$. We observe that if $r < \sqrt[3]{\frac{3V}{8\pi}}$, then $f'(r) < 0$, and if $r > \sqrt[3]{\frac{3V}{8\pi}}$, then $f'(r) > 0$. By Theorem 5-4c, $f\left(\sqrt[3]{\frac{3V}{8\pi}}\right)$ is a minimum. If $r = \sqrt[3]{\frac{3V}{8\pi}}$, then $\ell = 2\left(\frac{V}{9\pi}\right)^{1/3}$, so that $\frac{r}{\ell} = \frac{3}{4}$. The most economical dimensions are $r = \frac{3}{4}\ell$.



7. Find the length of the longest rod which can be carried horizontally around a corner from a corridor 10 ft. wide into one 5 ft. wide.



See Exercises 1-1, Number 10.

$L = f(\theta) = 5(2\csc\theta + \sec\theta)$

$f'(\theta) = 5(-2\csc\theta\cot\theta + \sec\theta\tan\theta)$

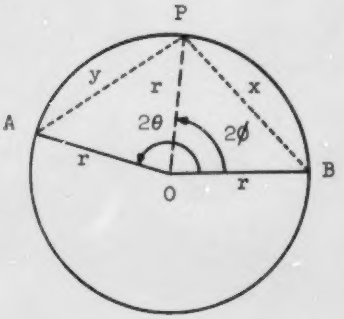
is zero if $\tan^3\theta = 2$ and

$\theta = \arctan 2^{1/3}$. For this value of θ,

$$L = f(\theta) = 5\left(2^{2/3}(2^{2/3} + 1)^{1/2} + (2^{2/3} + 1)^{1/2}\right)$$
$$= 5(2^{2/3} + 1)^{3/2}.$$

To see that this is a minimum value of $f(\theta)$, we observe that there is only one extremum in the interval $(0,\pi)$ and for θ near 0 and π in this interval, $f(\theta)$ is very large. So $L = f(\arctan 2^{1/3})$ is a minimum of f.

8. Find a point P on the arc AB such that the sum of the lengths of the chords AP and BP is a maximum$(\theta \leq \frac{\pi}{2})$.



Let angle POB = 2ϕ . Then angle POA = $2\theta - 2\phi$. If we let x equal length of chord PB and y equal length of chord AP , then since triangles POB and POA are isosceles, we have

$$x = 2r \sin \phi$$
$$y = 2r \sin(\theta - \phi) .$$

To maximize x + y let

$$f(\theta) = 2r\Big(\sin \phi + \sin(\theta - \phi)\Big) , \qquad 0 \leq \phi \leq \theta ,$$

whence

$$f'(\theta) = 2r\Big(\cos \phi - \cos(\theta - \phi)\Big) = 0$$

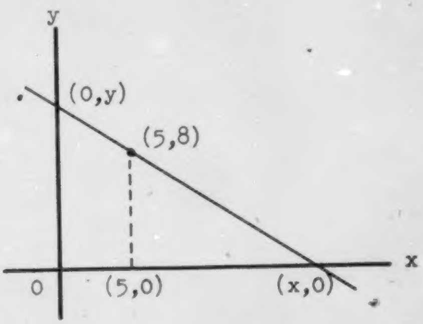
whence

$$\phi = \frac{\theta}{2} .$$

This occurs when P is the midpoint of arc AB .

One could also consider the family of (confocal) ellipses whose foci are at A and B . By symmetry, the largest ellipse intersecting the arc (minor) AB will be tangent to it at its midpoint.

9. Show how to determine a line, if possible, which passes through the point (5,8) such that the area of the triangle formed in the first quadrant is a positive number a . For what values of a is it impossible to construct such a triangle?



By similar triangles,

$$\frac{y}{x} = \frac{8}{x - 5}$$

or

$$y = \frac{8x}{x - 5} .$$

Let the area of the triangle be f(x) , then

$$f(x) = \frac{1}{2}xy = \frac{4x^2}{x - 5} , \quad x > 5 .$$

Then $f(x) = a$ only if the quadratic equation $4x^2 = a(x - 5)$ has real roots. Since

$$x = \frac{a \pm \sqrt{a(a - 80)}}{8},$$

a must be greater than or equal to 80 . This implies that the minimum value of

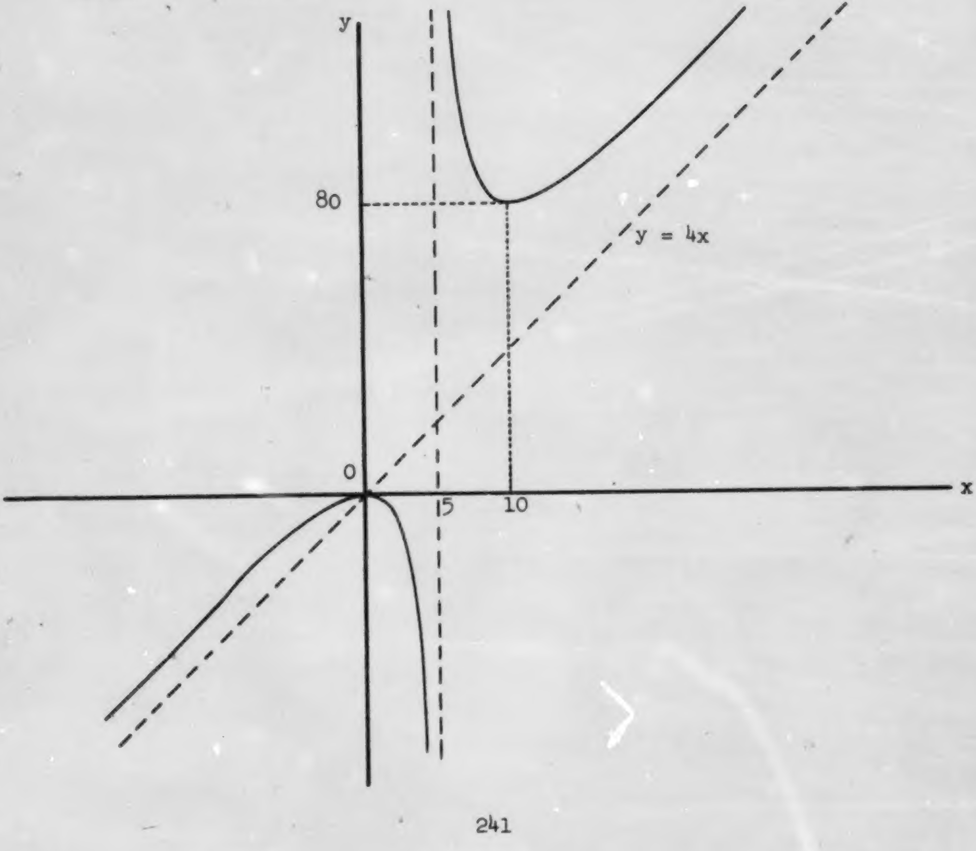
$$f(x) = \frac{4x^2}{x - 5}, \qquad x > 5$$

is 80 . Note that for $a > 80$, there are two triangles satisfying our conditions.

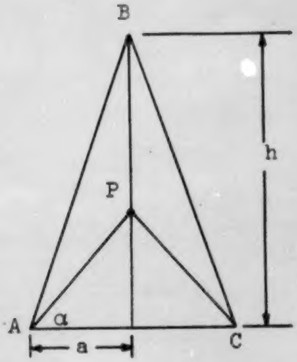
We can also obtain this result by setting

$$f'(x) = 0 = \frac{4x(x - 10)}{(x - 5)^2}.$$

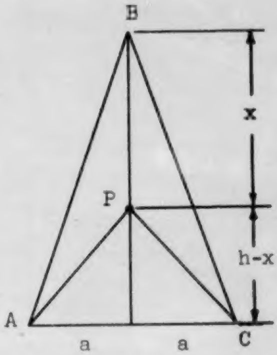
Thus $x = 10$ and $f(10) = 80$. A graph of f with the domain extended to all x is given.



10. **Find a point on the altitude of an isosceles triangle such that the sum of its distances from the vertices is the smallest possible.**



The sum of the distances, S, is the sum of the lengths of the segments AP, BP, and CP, where P is a point on the altitude of triangle ABC. $S = 2a \sec \alpha + h - a \tan \alpha$, where a represents $\frac{1}{2}$ the length of the base AC. $S = f(\alpha)$ so that $f'(\alpha) = a \sec \alpha\left(\frac{2 \sin \alpha - 1}{\cos \alpha}\right)$. We note that $f'(\alpha)$ is zero if and only if $\sin \alpha = \frac{1}{2}$, that is $\alpha = \frac{\pi}{6}$. If $\alpha > \frac{\pi}{6}$, then $f'(\alpha) > 0$ and if $\alpha < \frac{\pi}{6}$, then $f'(\alpha) < 0$, hence $f(\frac{\pi}{6})$ is a minimum. We note that this minimum is unobtainable if the base angles are less than $\frac{\pi}{6}$ radians, as P must remain in the interior of the triangle. If $\alpha \leq$ angle $A < \frac{\pi}{6}$ then the minimum of f is obtained at the endpoint of the interval, where $\alpha =$ angle A.



Alternative Solution:

$$S = f(x) = 2\sqrt{(h - x)^2 + a^2} + x$$

so that

$$f\ (x) = \frac{-2(h - x)}{\sqrt{(h - x)^2 + a^2}} + 1$$

is zero if

$$2(h - x) = \sqrt{(h - x)^2 + a^2}$$

or

$$(h - x) = \frac{a}{\sqrt{3}}.$$

This result agrees with that obtained above.

11. Let f be differentiable on a neighborhood of a point a for which $f'(a) = 0$. If $f'(x) \leq 0$ when $x < a$ and $f'(x) \geq 0$ when $x > a$ then $f(a)$ is a minimum. If $f'(x) \geq 0$ when $x < a$ and $f'(x) \leq 0$ when $x > a$ then $f(a)$ is a maximum. Give a proof.

We consider the case for $f(a)$ a minimum. The proof for $f(a)$ a maximum is similar. Let x be a point of a deleted neighborhood of a. By the Law of the Mean there is a number u such that $f(x) - f(a) = f'(u)(x - a)$ for $a < u < x$ and for $x < u < a$. From the hypothesis, whether $x < a$ or $x > a$, it follows that

$$f'(u)(x - a) \geq 0 .$$

We conclude that $f(x) \geq f(a)$ for all x in the neighborhood of a. Therefore $f(a)$ is a minimum.

12. Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). Suppose u is the one point in (a,b) where $f'(u) = 0$. Prove that if $f'(x)$ reverses sign in a neighborhood of u then $f(u)$ is the global extremum of f on $[a,b]$ appropriate to the sense of reversal.

By the hypothesis, $f'(x) = 0$ has only one solution, $x = u$. The derivative $f'(x)$ must have constant sign in each of the intervals (a,u), (u,b) or we could find another zero of the derivative by Exercises 5-3, Number 17. Since $f'(x)$ reverses sign in a neighborhood of u, either $f'(x) > 0$ for $x < u$, or $f'(x) > 0$ for $x > u$. We will consider the case where $f'(x) > 0$ for $x < u$ and $f'(x) < 0$ for $x > u$ (the proof for the other case is similar). By Theorem 5-4c, $f(u)$ is a maximum in a neighborhood of u. By the Law of the Mean, for the interval $[a,u]$, $f(a) - f(u) = f'(v)(a - u)$ for some v, $a < v < u$. Thus, in this case $f'(v)(a - u) < 0$ and $f(a) < f(u)$. In the same way, applying the Law of the Mean to the closed interval $[u,b]$ we can show that $f(b) < f(u)$. The only extrema on $[a,b]$ are at the endpoints or at points where $f'(x) = 0$. Since we have eliminated the endpoints as possible maxima, we conclude that $f(u)$ is a global maximum of f on $[a,b]$.

13. Given a function f such that $f(1) = f(2) = 4$, and such that $f''(x)$ exists and is positive throughout the interval $1 \leq x \leq 3$. What can you conclude about $f'(2.5)$? about $f(2.5)$? Prove your statements, stating whatever theorems you use in your proof. (Note: This statement of the problem differs from that in the text.)

$f(1) = f(2) = 4$.

Since f'' exists on the interval $[1,3]$, f' is continuous and differentiable on $[1,3]$ and f also is continuous and differentiable on $[1,3]$. By Rolle's Theorem, there is a number u , $1 < u < 2$, such that $f'(u) = 0$. Since $f''(x) > 0$ on the interval $[1,3]$, f' is increasing on $[1,3]$ and hence, for $u < x < 3$, $f'(x) > f'(u) = 0$. Thus $f'(2.5) > 0$. Since $f'(x) > 0$ for x in $(u,3)$, f is increasing in $[u,3]$; since $u < 2$, it follows that $f(2.5) > f(2) = 4$.

14. Let f be a differentiable function on (a,b) . Prove that the requirement that f be increasing is equivalent to the condition that $f'(x) \geq 0$ everywhere but that every interval contains points where $f'(x) > 0$.

Let us assume first that f increasing. If there were an entire interval on which $f'(x) = 0$ then by Corollary 1 to Theorem 5-4a it would follow that f is constant on that interval in contradiction to the assumption that f is strongly increasing. On the other hand, suppose that $f'(x) \geq 0$ but that every interval contains points where $f'(x) > 0$. Take any pair of points x_1 , x_2 in (a,b) with $x_1 < x_2$. We will show $f(x_1) < f(x_2)$. By hypothesis, there is a point u in (x_1,x_2) where $f'(u) > 0$. Since $f'(u) = \lim_{x \to u} \frac{f(x) - f(u)}{x - u}$ it follows by Lemma 3-4 that for some sufficiently small δ-neighborhood of u within (x_1,x_2)

$$\frac{f(x) - f(u)}{x - u} > 0 . \tag{1}$$

Choose particular values p , q in this δ-neighborhood so that $p < u$ and $q > u$. We then have $x_1 < p < u < q < x_2$. It follows from (1) that

$$f(p) < f(u) < f(q) . \tag{2}$$

However, under the assumption $f'(x) \geq 0$ in (a,b), we have from Theorem 5-4a that

$$f(x_1) \leq f(p) \quad \text{and} \quad f(q) \leq f(x_2) . \tag{3}$$

Combining the results of (2) and (3) we see that if $x_1 < x_2$ then $f(x_1) < f(x_2)$, i.e., that $f(x)$ is increasing.

15. Given that f is everywhere differentiable. If for all x such that $f'(x) \geq 0$, $f(x) \leq f(0)$, prove that $f(x) \leq f(0)$ for all $x > 0$.

Suppose there exists an x_0 such that $f(x_0) > f(0)$.

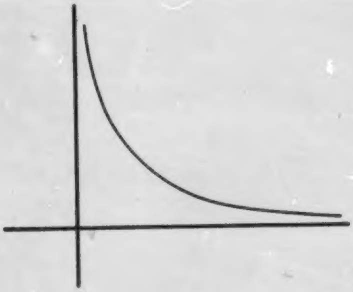
Let x_m be a point in $[0,x_0]$ where $f(x_m)$ is a maximum. Since $f(x_m) \geq f(x_0) > f(0)$, we have $x_m \neq 0$. If $x_m \neq x_0$ then $f'(x_m) = 0$ by Theorem 5-2a; if $x_m = x_0$ then $f'(x_m) \geq 0$ (if, on the contrary, $f'(x_m) < 0$ there would be a neighborhood of x_m wherein $f(x) > f(x_m)$ for $x < x_m$, as can be shown by the application of Lemma 3-4 in the preceding problem or by the result of Exercises 5-4, No. 17). In either case, from $f'(x_m) \geq 0$ we conclude by the hypothesis of the problem that $f(x_m) < f(0)$, a contradiction.

16. A function g is such that g'' is continuous and positive in the interval (p,q). What is the maximum number of roots of each of the equations $g(x) = 0$ and $g'(x) = 0$ in (p,q)? Prove your result and give some illustrative examples.

We have g'' is continuous and positive on (p,q). Then g' is continuous and increasing on (p,q) and thus can have at most one real root (else $g'(x_1) = g'(x_2) = 0$ and g' is not increasing).

If $g'(x) = 0$ for any value of x, say x_0, then $g'(x) > 0$ for $x > x_0$ and $g'(x) < 0$ for $x < x_0$ (g' is increasing). Then g is increasing for $x > x_0$ and decreasing for $x < x_0$ and can have at most two real roots. If $g'(x) = 0$ for no value of x, then either $g'(x) < 0$ for all x or $g'(x) > 0$ for all x, and $g(x)$ is either increasing or decreasing for all x and can have at most one real root.

Case 1.



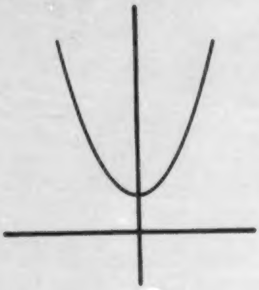
$g(x) = \frac{1}{x}$

$g''(x) > 0$

$g'(x)$ has no roots

$g(x)$ has no roots

Case 2.



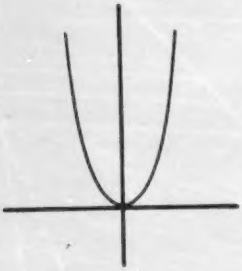
$g(x) = x^2 + 1$

$g''(x) > 0$

$g'(x)$ has one root

$g(x)$ has no root

Case 3.



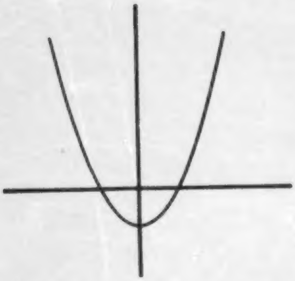
$g(x) = x^4 + x^2$

$g''(x) > 0$

$g'(x)$ has one root

$g(x)$ has one root

Case 4.



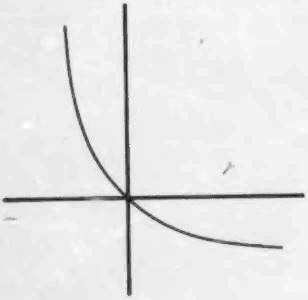
$g(x) = x^2 - 1$

$g''(x) > 0$

$g'(x)$ has one root

$g(x)$ has two roots

Case 5.



$g(x) = \frac{1}{x + 1} - 1$

$g''(x) > 0$

$g'(x)$ has no roots

$g(x)$ has one root

17. (a) If $f'(a) > 0$ show for values of x in a neighborhood of a that if $x > a$ then $f(x) > f(a)$, and if $x < a$ then $f(x) < f(a)$.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Thus by Lemma 3-4 there is a $\delta > 0$ such that $\frac{f(x) - f(a)}{x - a}$ is greater than zero for each x satisfying $0 < |x - a| < \delta$ if $f'(a) > 0$. Thus, if $x > a$ then $f(x) > f(a)$, and if $x < a$ then $f(x) < f(a)$.

(b) Give an example of a function f for which $f'(a) > 0$ but which is not increasing in any neighborhood of a, no matter how small.

One example is given by the function defined in Exercises 4-3a, Number 4. Another example, at $x = 0$, is the function defined by

$$f(x) = \begin{cases} x^2 \sin \frac{\pi}{x} + x, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

The product rule for differentiation does not apply here, so we fall back on the definition:

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} [x \sin \frac{\pi}{x} + 1] = 1 .$$

If $x \neq 0$ then $f'(x) = 1 - \pi \cos \frac{\pi}{x} + 2x \sin \frac{\pi}{x}$. If $x = \frac{1}{2n}$ then $\cos \frac{\pi}{x} = 1$ and if $x = \frac{1}{2n + 1}$ then $\cos \frac{\pi}{x} = -1$. Thus if $0 < x < \frac{\pi - 1}{4}$ then $f'(\frac{1}{2n}) < 0$ and $f'(\frac{1}{2n + 1}) > 0$. By Lemma 3-4 (f' is continuous at every point other than 0) f' is positive and therefore

f is increasing in a neighborhood of $\frac{1}{2n}$, and negative and therefore f is decreasing in a neighborhood of $\frac{1}{2n + 1}$. Thus every interval centered at 0 contains a subinterval where f is increasing and a subinterval where f is decreasing. Therefore f is not monotone in any interval which contains 0.

5-5. Applications of the Second Derivative.

The study of convex functions and convex sets is a beautiful subject in its own right. However, it is included here primarily as an aid in describing the appearance of the graph of a function. The generality with which we state the definition is more than we need for this course. Most of this discussion and corresponding exercises may appropriately be used as collateral study.

Definition 5-5 does not require that f be continuous. We can more fully appreciate the notion of convexity by seeing one implication of the requirement that a function be convex. Let us assume that f is flexed upward. It is geometrically evident and easily verified analytically that the difference quotient $\frac{f(x_0 + h) - f(x_0)}{h}$ is a nondecreasing function of h for fixed x . Moreover, $\frac{f(x_0 + h) - f(x_0)}{h} > \frac{f(x_0 - h) - f(x_0)}{-h}$ if $h > 0$. It then follows that the limit of $\frac{f(x_0 + h) - f(x_0)}{h}$ as h approaches 0 either through positive values or through negative values exists. Thus, a convex function must have right and left-sided derivatives everywhere. Contrast this kind of behavior with the bizarre behavior permitted a continuous function, say the Weierstrass function.

Solutions Exercises 5-5

Note: Number 14 is used in Numbers 15, 22-24. Numbers 21-23 should be assigned (as a block) together.

1. For each of the following functions, locate and characterize all extrema and state the intervals on which the function is increasing (decreasing). On what intervals is the graph flexed upward? downward?

   (a) $f : x \longrightarrow x^2 + x^{-2}$ (c) $f : x \longrightarrow x^{1/2} + x^{-1/2}$

   (b) $f : x \longrightarrow \frac{2}{x^2} + \frac{1}{x}$ (d) $f : x \longrightarrow x^3 + ax^2 + bx + c$

(a) $f(x) = x^2 + x^{-2}$

$$f'(x) = 2x - 2x^{-3} = \frac{2(x^4 - 1)}{x^3}$$

$$f''(x) = 2 + 6x^{-4} > 0 \quad \text{for} \quad x \neq 0 .$$

Thus, $f(1)$ is a local minimum of f, and $f(-1)$ is a local minimum of f. The function f is decreasing on the intervals $(0,1)$ and $x < -1$, and increasing on the intervals $(-1,0)$ and $x > 1$. The function f is flexed upward for all x in its domain, that is, for $x \neq 0$.

(b) $f(x) = \frac{2}{x^2} + \frac{1}{x}$

$$f'(x) = -\frac{1}{x^3}(4 + x)$$

$$f''(x) = \frac{2}{x^4}(6 + x)$$

$$f''(-4) > 0 .$$

Thus $f(-4)$ is a local minimum of f. The function f is decreasing for $x < -4$ and for $x > 0$, and is increasing on the interval $(-4,0)$. The function f is flexed upward on the interval $(-6,0)$ and for $x > 0$, downward for $x < -6$.

(c) $f(x) = x^{1/2} + x^{-1/2}$

$$f'(x) = \frac{x - 1}{2x^{3/2}}$$

$$f''(x) = \frac{3 - x}{4x^{5/2}}$$

$$f''(1) > 0 .$$

Thus $f(1)$ is a local minimum of f. The function f is decreasing on the interval $(0,1)$ and is increasing for $x > 1$. The function f is flexed upward on the interval $(0,3)$ and downward for $x > 3$.

(d) $f(x) = x^3 + ax^2 + bx + c$

$f'(x) = 3x^2 + 2ax + b$.

Every cubic is centrosymmetric about its point of inflection (see TC2-2, and TC2-3) and has either no maxima or minima (e.g., $f(x) = x^3$) or a maximum and minimum spaced equally apart from the center of symmetry.

For brevity, we assume $a , b > 0$. The solution for other cases is similar.

(i) If the discriminant of $f'(x) > 0$, then f will have a maximum, minimum and point of inflection. f will be flexed downward for $x < \frac{-a}{3}$, upward for $x > \frac{-a}{3}$, increasing for $x > \frac{-a + \sqrt{a^2 - 3b}}{3}$ and for $x < \frac{-a - \sqrt{a^2 - 3b}}{3}$, decreasing for $\frac{-a - \sqrt{a^2 - 3b}}{3} < x < \frac{-a + \sqrt{a^2 - 3b}}{3}$.

(ii) If the discriminant of $f' \leq 0$, f will have only a point of inflection at $x = \frac{-a}{3}$. No maximum or minimum; f increasing for all x .

2. Show that the graph of the function
$$f : x \longrightarrow 3 \sin 2x + 5 \cos 2x$$
is flexed upward when $f(x) < 0$ and flexed downward when $f(x) > 0$.

$$\begin{aligned} f(x) &= 3 \sin 2x + 5 \cos 2x \\ f'(x) &= 6 \cos 2x - 10 \sin 2x \\ f''(x) &= -12 \sin 2x - 20 \cos 2x \\ &= -4\, f(x) \ . \end{aligned}$$

The conclusion follows at once by Theorem -5b.

3. Find and characterize the extrema of the function
$$f : x \longrightarrow x \sin x + \cos x$$
on the closed interval $[0,\pi]$. On what intervals is the graph of the function flexed downward? upward?

$$\begin{aligned} f(x) &= x \sin x + \cos x \\ f'(x) &= x \cos x \\ f''(x) &= \cos x - x \sin x \ . \end{aligned}$$

$f(0) = 1$ and $f(\pi) = -1$. $f'(x) = 0$ when $x = \frac{\pi}{2}$ or $x = 0$. $f(\frac{\pi}{2}) = \frac{\pi}{2}$. So $f(\frac{\pi}{2})$ is the maximum of f on $[0,\pi]$ and $f(\pi)$ the minimum. $f''(x) = 0$ when $x = \cot x$, or $x = x_0 \approx .86$. This is the only zero of f'' on $[0,\pi]$. Then since $f''(0) = 1 > 0$, $f''(x) \geq 0$ on $[0,x_0]$ by the Intermediate Value Theorem and the graph of f is flexed upward on that interval. Similarly, f is flexed downward on $[x_0,\pi]$.

4. Assume that the function f has a local maximum (or minimum) at $x = a$, where $f'(a) = 0$, and $f''(a) \neq 0$. Determine conditions on a function g , assumed differentiable, such that $x \longrightarrow gf(x)$ also has a local maximum (or minimum) at $x = a$.

f has a local maximum at a. Thus, $f'(a) = 0$. Also we have $f''(a) < 0$. To guarantee that $gf(x)$ has a local maximum at a , we require $[gf(a)]' = 0$ and $[gf(a)]'' < 0$.

$$[gf(x)]' = g'f(x) \cdot f'(x) = 0$$

$$[gf(x)]'' = g''f(x) \cdot f'(x)^2 + g'f(x) \cdot f''(x) .$$

Since $f''(a) < 0$ and $f'(a) = 0$, $[gf(a)]'' < 0$ only if $g'f(a) > 0$.

If g is an increasing function of f , and f has a local maximum of a then the conclusion that gf has a local maximum at a follows easily without calculus.

We use similar reasoning for the minimum.

5. Use Theorem 5-5a to locate and classify all extrema of the function
$$f : x \longrightarrow \sin x(1 + \cos x)$$
on the closed interval $[-\pi,\pi]$. On what intervals is the graph of the function flexed downward? upward?

$f(x) = \sin x(1 + \cos x) = \sin x + \frac{\sin 2x}{2}$

$f'(x) = \cos x + \cos 2x = (2 \cos x - 1)(\cos x + 1)$

$f''(x) = -\sin x - 2 \sin 2x = -\sin x(1 + 4 \cos x)$

Since $\cos(\pm \frac{\pi}{3}) = \frac{1}{2}$ and $f'(\pm \frac{\pi}{3}) = 0$, we obtain $f(\frac{\pi}{3}) = \frac{3\sqrt{3}}{4}$ as a maximum value of f on $[-\pi,\pi]$ and $f(-\frac{\pi}{3}) = \frac{-3\sqrt{3}}{4}$ as a minimum value. The graph of f is flexed upward on the intervals $(-\pi,-x_0)$ and $(0,x_0)$, where $\cos x_0 = \frac{1}{4}$ and $x_0 \approx 1.82$, approximately. The graph of f is flexed downward on the intervals $(-x_0,0)$ and (x_0,π) .

6. Let $f(x) = x - \sin x$. Does f have any extrema? Justify your answer.

$f(x) = x - \sin x$

$f'(x) = 1 - \cos x$

$f'(x) = 0$ if $\cos x = 1$; that is, if $x = 2n\pi$, n an integer, otherwise $f'(x) > 0$. By Theorem 5-4a the function f is weakly increasing on any neighborhood of $x = 2n\pi$, n an integer, and hence f has no extrema.

7. Let $f(x) = \frac{x}{\sin x}$. Does f have any extrema in the open interval $(0,\frac{\pi}{2})$? Justify your answer.

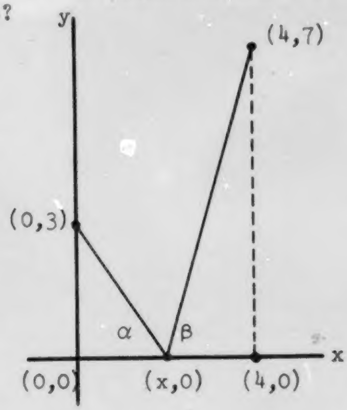
$$f(x) = \frac{x}{\sin x}$$

$$f'(x) = \frac{\sin x - x \cos x}{\sin^2 x}$$

If $f'(x) = 0$, then $\sin x - x \cos x = 0$, or equivalently, $\tan x = x$. However, in the interval $(0,\frac{\pi}{2})$, $\tan x = x$ has no solution; hence f has no extrema in the open interval $(0,\frac{\pi}{2})$. To see that $\tan x = x$ has no solution in $(0,\frac{\pi}{2})$ consider $g(x) = \tan x - x$.

We have $g'(x) = \sec^2 x - 1$ so that $g'(x) > 0$ for all x in $(0,\frac{\pi}{2})$. Since $g(0) = 0$ it follows that $g(x) > 0$ for all x in $(0,\frac{\pi}{2})$, by Theorem 5-4a.

8. At what point of the positive x-axis is the angle subtended by the two points $(0,3)$ and $(4,7)$ greatest?



See figure at right: $\tan \alpha = \frac{3}{x}$ and $\tan \beta = \frac{7}{4 - x}$ so that

$$\tan(\alpha + \beta) = \frac{\frac{3}{x} + \frac{7}{4 - x}}{1 - \frac{3}{x} \cdot \frac{7}{4 - x}} = \frac{4(3 + x)}{x(4 - x) - 21} .$$

It is sufficient to find x so that $\alpha + \beta$ is a minimum or that

$\tan(\alpha + \beta)$ is a minimum (see No. 4).

$$D_x \tan(\alpha + \beta) = \frac{4(x^2 + 6x - 33)}{(4x - x^2 - 21)^2} = F(x)$$

$$F(x) = 0 \text{ if } x = \sqrt{42} - 3 .$$

If x is in the interval $(0, \sqrt{42} - 3)$ then $F(x) < 0$, and if x is greater than $\sqrt{42} - 3$ then $F(x) > 0$. Thus, $\tan(\alpha + \beta)$ is a minimum if $x = \sqrt{42} - 3$, by Theorem 5-4c. It follows that the angle subtended by the points $(0,3)$ and $(4,7)$ is the greatest at the point $x = \sqrt{42} - 3$.

9. Suppose that $f'(a) = f''(a) = \dots = f^{(n-1)}(a) = 0$ but that $f^{(n)}(a) \neq 0$. Determine whether $f(a)$ is a local extremum and if it is, which kind. (Hint: consider separately the cases n even and n odd.)

Let $f'(a) = f''(a) = \dots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$, then

(i) whenever n is even and $f^{(n)}(a) > 0$, $f(a)$ is a local minimum value of f.

(ii) whenever n is even and $f^{(n)}(a) < 0$, $f(a)$ is a local maximum.

(iii) whenever n is odd, $f(a)$ is not a local extremum.

The proof of (i) is typical: From the proof of Theorem 5-5a and from Theorem 5-4c we know that $f^{(n-2)}(a)$ is an isolated minimum of $f^{(n-2)}$. In a deleted neighborhood of a we have $f^{(n-2)}(x) > f^{(n-2)}(a) \geq 0$. We conclude that the graph of $f^{(n-4)}$ is flexed upward in the neighborhood of a and hence that $f^{(n-4)}(a)$ is a minimum of $f^{(n-4)}$. Iterating this argument we obtain the desired result.

10. Prove if the graph of f is flexed downward on an interval I, that the set D of points under the graph is a convex set.

We have

$$D = \{(x,y) : x \in I \text{ and } y \leq f(x)\}$$

and wish to prove that D is convex. Now D is convex if it contains with every pair of points of D, the entire line segment joining them.

Proof by Contradiction.

Assume that D is not convex. Then there exist two points $\alpha = (x_a, y_a)$ and $\beta = (x_b, y_b)$ such that the segment joining α and β is not entirely in D.

Since $\alpha \in D$ and $\beta \in D$ there is at least one point (t,y_t), where $x_a < t < x_b$, on the segment which is not in D. This means that $y_t - f(t) > 0$. It follows by continuity that there is a neighborhood of $x = t$ where $y - f(t) > 0$. Hence the segment joining α and β must cut the graph of f in two points, say $(u,f(u))$ and $(v,f(v))$, where $x_a \leq u < t$ and $t < v \leq x_b$, by the hypothesis and the Intermediate Value Theorem (Theorem 3-7a). The segment joining $(u,f(u))$ and $(v,f(v))$ is a chord to the graph of f, and hence no point of the chord is above the graph of f since f is convex, by Definition 5-5. It follows that $y_t \leq f(t)$. We have a contradiction. Therefore D is convex.

Alternative Proof.

Consider p, q, r, any points of I with $p < q < r$. Also, consider D, the set of points under the downwardly flexed graph of the function f, i.e.,

$$D = \{(x,y) : x \in I \text{ and } y \leq f(x)\} .$$

Let (p,y_p) and (r,y_r) be any points in D corresponding to p and r. The chord joining (p,y_p) and (r,y_r) has equation

$$y = y_p + \frac{y_r - y_p}{r - p}(x - p)$$

or

$$\frac{y - y_p}{x - p} = \frac{y_r - y_p}{r - p} .$$

The condition that D is a convex set requires that the point (q,y_q) of the chord not be above the point $(q,f(a))$ on the arc of f, or that $f(q) \geq y_q$.

11. Let f be continuous on an interval I. Prove that a necessary and sufficient condition that the graph of f be flexed downward on an interval I is that the slope of the chord joining a point $(x,f(x))$ to a fixed point $(a,f(a))$ is a weakly decreasing function of x. (It is understood that x and a lie in I.) See Figure 5-5a.

Note: The condition is required for every point a in I.

First we shall prove that if the graph of f is flexed downward then the slope of chords through any fixed point a in I is a weakly decreasing function.

Proof. Let a be any fixed point in I. Equations of chords through $\big(a,f(a)\big)$ and $\big(q,f(q)\big)$, for other points $q \neq a$ in I, will be of the form

$$y = g(x) = f(a) + (x - a)\,\frac{f(q) - f(a)}{q - a} .$$

By Definition 5-5, the graph of f is flexed downward only if $f(p) \geq g(p)$ for all p between a and q, i.e., for all p such that $a < p < q$ or $q < p < a$. In either case we have

$$f(p) \geq g(p) \geq f(a) + (x - a)\,\frac{f(q) - f(a)}{q - a} .$$

If $p > a$, then $q > p$, and this becomes

$$\frac{f(p) - f(a)}{p - a} \geq \frac{f(q) - f(a)}{q - a} .$$

If $p < a$, then $p > q$, and we have

$$\frac{f(p) - f(a)}{p - a} \leq \frac{f(q) - f(a)}{q - a} .$$

In both cases we have shown the slope function

$$x \longrightarrow \frac{f(x) - f(a)}{x - a}$$

to be weakly decreasing.

Now we shall prove the converse. By Definition 5-5, we need to show that if the slope function for chords through any point a in I is weakly decreasing, then if $y = g(x)$ is the (linear) equation of an arbitrary chord whose endpoints are $\big(b,f(b)\big)$ and $\big(c,f(c)\big)$, b and c in I, then $f(p) \geq g(p)$ for any p between b and c.

Proof. Let $b < p < c$. Then by hypothesis,

$$\frac{f(p) - f(b)}{p - b} \geq \frac{f(c) - f(b)}{c - b} .$$

The equation of the chord through $\big(b,f(b)\big)$ and $\big(c,f(c)\big)$ is

$$y = g(x) = f(b) + (x - b)\,\frac{f(c) - f(b)}{c - b} .$$

By definition of g,

$$\frac{g(x) - g(b)}{x - b} = \frac{f(c) - f(b)}{c - b} ,$$

so that

$$\frac{g(x) - f(b)}{x - b} \leq \frac{f(p) - f(b)}{p - b} ,$$

for all x in I . In particular, taking $x = p$, we obtain

$$g(p) \leq f(p) \text{ , q.e.d.}$$

12. (a) Let f be differentiable and its graph be flexed downward on an interval I . Prove that the function

$$\phi(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a} , & x \neq a \\ f'(a) , & x = a \end{cases}$$

is weakly decreasing, where the fixed point a is any interior point of I .

We have already shown in Number 11 that ϕ is weakly decreasing on I with the point a deleted. We now show that ϕ is weakly decreasing on the entire interval.

Proof by Contradiction.
Suppose that $t > a$ and $\phi(a) < \phi(t)$.
Then for $\epsilon = \phi(t) - \phi(a)$ there exists a $\delta > 0$ such that

$$|\phi(x) - \phi(a)| < \epsilon \text{ whenever } |x - a| < \delta .$$

Choose a value x between a and t within the δ neighborhood of a . We then have $x < t$, but

$$\phi(x) - \phi(a) \leq |\phi(x) - \phi(a)| < \phi(t) - \phi(a)$$

whence $\phi(x) < \phi(t)$, in contradiction to the weakly decreasing property of ϕ on the deleted interval.
A similar argument for $t < a$ completes the proof.

(b) From the result of (a) , prove that a necessary and sufficient condition that the graph of f be flexed downward on I is that f' be weakly decreasing.

We first prove that if the graph of f is flexed downward on I , then f' is weakly decreasing on I .

Proof. Let a , b be points of I such that $a < b$.
The function ϕ defined in part (a) and the function ψ defined by

$$\psi(x) = \begin{cases} \dfrac{f(x) - f(b)}{x - b} \ , & x \neq b \\ \\ f'(b) \ , & x = b \end{cases}$$

are both decreasing on I . Since $\phi(b) = \psi(a)$, $\phi(a) \geq \phi(b)$, and $\psi(a) \geq \psi(b)$, it follows that

$$f'(a) \geq \phi(b) \geq \psi(a) \geq f'(b) \ .$$

In other words, f' is weakly decreasing on I .

Conversely, we show if f' is weakly decreasing on I , then the graph of f is flexed downward on I .

Proof. For any points p , q , r of I with $p < q < r$ we have by the Law of the Mean

$$\frac{f(q) - f(p)}{q - p} = f'(t) \ , \text{ where } \ p < t < q$$

and
$$\frac{f(r) - f(q)}{r - q} = f'(u) \ , \text{ where } \ q < u < r \ .$$

Since $t < u$, and since f' is weakly decreasing (by hypothesis) we have $f'(t) \geq f'(u)$. It follows that

$$\frac{f(q) - f(p)}{q - p} \geq \frac{f(r) - f(q)}{r - q} \ .$$

This inequality can be interpreted geometrically as a statement that the slope of the chord joining the fixed point $(q,f(q))$ to any point $(x,f(x))$ x I , is a decreasing function of x on I . It follows by Number 11 that the graph of f is flexed downward.

13. Prove Theorem 5-5b.

If f has a second derivative f'' and if $f''(x) \leq 0$ on the open interval (a,b) , then the graph of f is flexed downward on I .

Proof. If $f''(x) \leq 0$, then f' is a weakly decreasing function on (a,b) by Theorem 5-4a.

It follows that the graph of f is flexed downward on (a,b) , by the results of Exercises 5-5, Number 11(b).

Conversely, if f is convex on (a,b) and has a second derivative there, then $f''(x) \leq 0$.

Alternative Proof. If f is convex on (a,b) then f' is weakly decreasing on (a,b) by the results of Number 11(b) above. Since f is twice differentiable on (a,b), $f''(x) \leq 0$ by Theorem 5-4a, since f' is weakly increasing on (a,b).

14. (a) Let x and y be two points on an interval I in the domain of a function f. Show that a point is on the chord joining the points $(x,f(x))$ and $(y,f(y))$ on the graph of f if and only if its coordinates are
$$\big(\theta x + (1 - \theta)y \, , \theta f(x) + (1 - \theta)f(y)\big)$$
for some θ such that $0 \leq \theta \leq 1$.

If $A(a',a)$ and $B(b',b)$ are points in a plane, then (x,y) is on the line $\overleftrightarrow{AB}$ if and only if

$$\frac{y - b}{x - a} = \frac{b' - b}{a' - a} \quad \text{or} \quad y = \frac{b' - b}{a' - a}(x - a) + b \, .$$

If we further require that (x,y) be on the segment $\overline{AB}$, then x must be between a and a', inclusive. Then $\frac{x - a}{a' - a}$ takes on all values between 0 and 1, inclusive. If we let $\theta = \frac{x - a}{a' - a}$

then $\qquad x = (a' - a)\theta + a$

and $\qquad y = (b' - b)\theta + b$

or $\qquad x = \theta a' + (1 - \theta)a$,

and $\qquad y = \theta b' + (1 - \theta)b$,

which was to be shown.

Since $\frac{x - a}{a' - a} = \theta$ takes on all values between 0 and 1 for $x : a < x < a'$ and $y : b < y < b'$, the converse is also true and all points $\big(\theta a + (1 - \theta)a' \, , \theta b + (1 - \theta)b'\big)$ are on $\overline{AB}$.

(b) Show that Definition 5-5 asserts that f is flexed upward on I if and only if for all x and y in I and all $\theta : 0 \leq \theta \leq 1$,
$$f\big(\theta x + (1 - \theta)y\big) \leq \theta f(x) + (1 - \theta)f(y) \, .$$

Given a function f defined on an interval I, any point (x,y) on a chord of the graph of f will be of the form

$$x = \theta a + (1 - \theta)a' \, ,$$
$$y = \theta f(a) + (1 - \theta)f(a') \, ,$$

where $\theta : 0 \leq \theta \leq 1$ and a', a are any two points on I. If we require that f is convex on I , and that additionally, the graph of f is flexed upward, then, by Definition 5-5, all chords lie above their corresponding arcs, or

$$f\big(\theta a + (1 - \theta)a'\big) \leq \theta f(a) + (1 - \theta)f(a') .$$

This equation, then, is just the analytic characterization of f taken to be convex with graph flexed upward. Similarly, if the graph of f is flexed downward,

$$f\big(\theta a + (1 - \theta)a'\big) \geq \theta f(a) + (1 - \theta)f(a') .$$

(c) Use (b) to show that the graphs of the following functions are flexed upward.

(i) $f : x \longrightarrow ax + b.$

(ii) $f : x \longrightarrow x^2.$

(iii) $f : x \longrightarrow -\sqrt{x}$.

(i) $f(x) = ax + b$

$$\begin{aligned} f\big(\theta x + (1 - \theta)y\big) &= a\big(\theta x + (1 - \theta)y\big) + b \\ &= \theta(ax + b) + (1 - \theta)(ay + b) \\ &= \theta f(x) + (1 - \theta)f(y) . \end{aligned}$$

(ii) $f(x) = x^2$

$$f\big(\theta x + (1 - \theta)y\big) = \big(\theta x + (1 - \theta)y\big)^2$$

We want to show

$$\big(\theta x + (1 - \theta)y\big)^2 \leq \theta x^2 + (1 - \theta)y^2 .$$

$$\begin{aligned} \big(\theta x + (1 - \theta)y\big)^2 - \theta x^2 + (1 - \theta)y^2 & \\ &= (\theta^2 - \theta)x^2 - 2(\theta^2 - \theta)xy + \big((1 - \theta)^2 - (1 - \theta)\big)y^2 \\ &= (\theta^2 - \theta)(x - y)^2 \leq 0 \end{aligned}$$

since $(x - y)^2 > 0$ and $\theta^2 - \theta \leq 0$ $(0 \leq \theta \leq 1)$.

(iii) $f(x) = -\sqrt{x}$

$$f\big(\theta x + (1 - \theta)y\big) = \sqrt{\theta x + (1 - \theta)y}$$

We want to show

$$\sqrt{\theta x + (1 - \theta)y} \geq \theta\sqrt{x} + (1 - \theta)\sqrt{y} .$$

If this is true, then

$$\theta x + (1 - \theta)y \geq \theta^2 x + 2\theta(1 - \theta)\sqrt{xy} + (1 - \theta)^2 y .$$

Since both sides are nonnegative, then

$$(\theta - \theta^2)x - 2(\theta - \theta^2)\sqrt{xy} + (\theta - \theta^2)y \geq 0$$

or $(\theta - \theta^2)(\sqrt{x} - \sqrt{y})^2 \geq 0$, which is a true statement.

Starting from this fact and reversing the steps taken above, we obtain the desired result.

15. (a) Derive the following property of convex functions. If the graph of f is flexed downward on an interval I , then for all points a , b in I and any positive numbers p , q

$$f\left(\frac{pa + qb}{p + q}\right) \geq \frac{pf(a) + qf(b)}{p + q} .$$

In words, the function value of a weighted average is less than the weighted average of the function values.

We have for f that

$$f\big(\theta a + (1 - \theta)b\big) \geq \theta f(a) + (1 - \theta)f(b)$$

for all $\theta : 0 \leq \theta \leq 1$.

But $$0 \leq \frac{p}{p + q} \leq 1$$

and $$1 - \frac{p}{p + q} = \frac{q}{p + q} .$$

Setting $\theta = \frac{p}{p + q}$, we have

$$f\left(\frac{pa}{p + q} + \frac{qb}{p + q}\right) = f\left(\frac{pa + qb}{p + q}\right) \geq \frac{pf(a) + qf(b)}{p + q} .$$

(b) Prove that this property is sufficient for downward flexure.

Since $f\left(\frac{pa + qb}{p + q}\right) \geq \frac{pf(a) + qf(b)}{p + q}$ for all p , q positive, $\frac{p}{p + q}$ takes on all values between 0 and 1 exclusive. Then setting $\theta = \frac{p}{p + q}$, $f\big(\theta a + (1 - \theta)b\big) \geq \theta f(a) + (1 - \theta)f(b)$ for θ: $0 < \theta < 1$. For $\theta = 0$, $f(a) \geq f(a)$ and for $\theta = 1$, $f(b) \geq f(b)$. Then by Number 14, the graph of f is flexed downward. Together (a) and (b) demonstrate necessity and sufficiency. The criterion then is an alternative way of characterizing convexity.

16. Prove that if f is continuous, then a necessary and sufficient condition for its graph to be flexed downward is that

$$f\left(\frac{a + b}{2}\right) \geq \frac{f(a) + f(b)}{2} .$$

To prove necessity, use results established in Exercises 5-5, Number 15, and set $p = q = 1$.

To prove sufficiency we must show that if f is continuous and

$$f\left(\frac{a+b}{2}\right) \geq \frac{f(a) + f(b)}{2} \tag{1}$$

for all x, then the graph of f is flexed downward. We observe from (1) that

$$\begin{aligned} f\left(\frac{x_1 + x_2 + x_3 + x_4}{4}\right) &\geq \frac{1}{2}\left(f\left(\frac{x_1 + x_2}{2}\right) + f\left(\frac{x_3 + x_4}{2}\right)\right) \\ &\geq \frac{1}{4}\Big(f(x_1) + f(x_2) + f(x_3) + f(x_4)\Big) . \end{aligned}$$

Doubling the number of points repeatedly we obtain

$$f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right) \geq \frac{1}{n}\Big(f(x_1) + \ldots + f(x_n)\Big) \tag{2}$$

where n is a power of 2.

Now, if p and q are nonnegative integers with $p + q = n$ in (2) we obtain

$$f\left(\frac{pa + qb}{p + q}\right) \geq \frac{pf(a) + qf(b)}{p + q} \tag{3}$$

To prove (3) for all real values of p and q, not just nonnegative values which add up to a power of 2, we must use the continuity of f. Convexity will then follow from the result of Exercise 15.

Setting $\theta = \frac{p}{p+q}$ we rewrite (3) in the form

$$f\Big(\theta a + (1 - \theta)b\Big) \geq \theta f(a) + (1 - \theta)f(b) \tag{4}$$

where $\theta = \frac{p}{p+q}$ is any rational number with denominator a power of 2 satisfying $0 \leq \theta \leq 1$. For any value x_0 in $[a,b]$ we put

$$x_0 = ra + (1 - r)b$$

where r is now a real number satisfying $0 \leq r \leq 1$. Since f is continuous at x_0, for any positive ϵ we may choose a $\delta > 0$ so that $|x - x_0| < \delta$ insures $|f(x) - f(x_0)| < \epsilon$. We put $x = \theta a + (1 - \theta)b$ where θ is rational and has a power of 2 for its denominator. We have

$$|x - x_0| = |(\theta - r)(a - b)| = (b - a)|\theta - r| < \delta$$

provided $|\theta - r| < \frac{\delta}{b-a}$. It follows that $-\epsilon < f(x) - f(x_0) < \epsilon$.

At the same time

$$\begin{aligned} &|\theta f(a) + (1-\theta)f(b) - rf(a) - (1-r)f(b)| \\ &\quad = |\theta - r| \cdot |f(b) - f(a)| \\ &\quad < \frac{\delta|f(b) - f(a)|}{b-a}. \end{aligned}$$

We have

$$\begin{aligned} f(x_0) > f(x) - \epsilon &\geq \theta f(a) + (1-\theta)f(b) - \epsilon \\ &\geq rf(a) + (1-r)f(b) - \epsilon - \frac{\delta|f(b) - f(a)|}{b-a} \end{aligned}$$

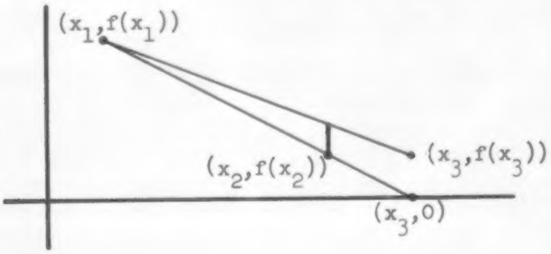
Given any $\epsilon^* > 0$, then, we take $\epsilon = \frac{\epsilon^*}{2}$ and $\delta^* \leq \delta$ but sufficiently small that $\frac{\delta^*|f(b) - f(a)|}{b-a} < \frac{\epsilon^*}{2}$ and choose an approximation θ to r such that $|\theta - r| < \frac{\delta^*}{b-a}$. We then have $f(ra + (1-r)b) \geq rf(a) + (1-r)f(b) - \epsilon^*$ for each positive ϵ^*. This can only be true if

$$f(ra + (1-r)b) \geq rf(a) + (1-r)f(b).$$

We have extended Equation (4) to real values and the convexity of f follows.

17. The graph of a function f is flexed downward and $f(x)$ is positive for all x. Show that f is a constant function. Do not assume f' exists.

Solution. Suppose f is nonconstant. Then there exist $x = x_1$ and $x = x_2$ such that $x_1 < x_2$ and $f(x_1) \neq f(x_2)$, say $f(x_1) > f(x_2)$. Now the line through $(x_1, f(x_1))$, $(x_2, f(x_2))$ intersects the x-axis at some point $x_3 > x_2$. We claim that $f(x_3) \leq 0$, which would contradict $f(x)$ positive. For if $f(x_3) > 0$, then the point $(x_2, f(x_2))$ would lie below the line segment connecting $(x_1, f(x_1))$ and $(x_3, f(x_3))$, contradicting f flexed downward. So the assumption of f nonconstant has led to a contradiction.



18. Under what circumstances will the graph of a function f and its inverse both be flexed downward? One flexed downward and the other upward? Answer this question both with and without calculus.

Without calculus: Let [a,b] be an interval in the domain of f. If the graph of f on [a,b] is flexed upward, it is interior to the angle of (a,f(a)) formed by the ray to (b,f(b)) with the ray going vertically upward. Reflection in the line $y = x$ takes the upward ray into a ray going horizontally to the right and the other ray in the upper half-plane. If the angle was acute (f increasing) the reflected graph lies in the first quadrant below the chord. If the angle was obtuse (f decreasing) the reflected graph lies in the second quadrant above the chord. The same argument can be made algebraically.

With calculus: Assume f' and f'' exist. Then g' and g'' exist, where g is the inverse of f. We have

$$1 = [gf(x)]' = g'f(x) \cdot f'(x)$$

or

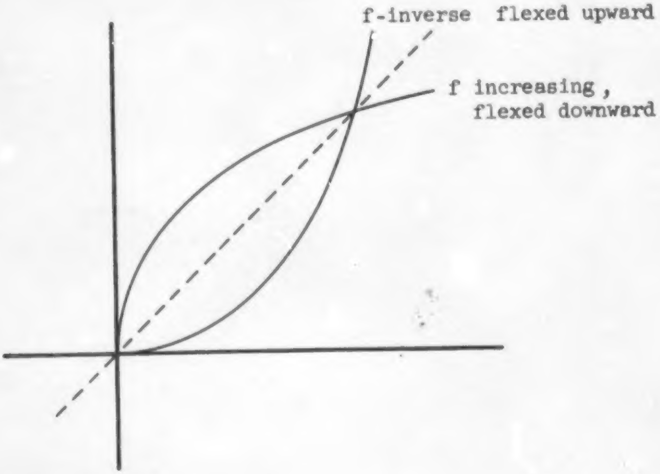
$$g'f(x) = \frac{1}{f'(x)}$$

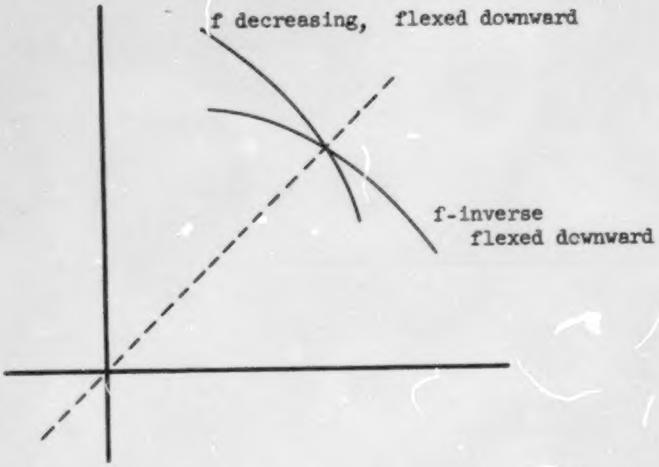
$$g''f(x) \cdot f'(x) = \frac{-f''(x)}{[f'(x)]^2}$$

or

$$g''f(x) = \frac{-f''(x)}{[f'(x)]^3} .$$

If $f''(x) < 0$, then $g''f(x) > 0$ if $f'(x) > 0$; $g''f(x) < 0$ if $f'(x) < 0$.




f decreasing, flexed downward
f-inverse
flexed downward

19. If either of $D^2xF(x)$ or $D^2F(\frac{1}{x})$ is of one sign for $x > 0$, show that the other one has the same sign. Interpret geometrically and illustrate by several examples.

Define f and g by $f : x \longrightarrow xF(x)$ and $g : x \longrightarrow F(\frac{1}{x})$. We have

$$f''(x) = xF''(x) + 2F'(x)$$

and

$$g''(x) = \frac{F''(\frac{1}{x})}{x^4} + \frac{2F'(\frac{1}{x})}{x^3} = x^3 f''(\frac{1}{x}) .$$

Since $x^3 > 0$ for $x > 0$, the result follows immediately. Two routine examples follow.

(i)

$$F(x) = x^3 + x > 0$$

$$F'(x) = 3x^2 + 1 > 0$$

$$F''(x) = 6x > 0$$

$$F'(\frac{1}{x}) = \frac{3}{x^2} + 1 > 0$$

$$F''(\frac{1}{x}) = \frac{6}{x} .$$

$$xF''(x) + 2F'(x) > 0 \text{ for all } x > 0$$

$$F''(\frac{1}{x})(\frac{1}{x^4}) + F'(\frac{1}{x})\frac{2}{x^3} > 0 \text{ for all } x > 0 .$$

(ii) $F(x) = \frac{x^4}{12} - \frac{x^3}{3} + x^2$.

$$F'(x) = \frac{x^3}{3} - x^2 + 2x \ ; \ F'(\tfrac{1}{x}) = \frac{1}{3x^3} - \frac{1}{x^2} + \frac{2}{x}$$

$$F''(x) = x^2 - 2x + 2 \ ; \ F''(\tfrac{1}{x}) = \frac{1}{x^2} - \frac{2}{x} + 2 \ .$$

$$xF''(x) + 2F'(x) = x^3 - 2x^2 + 2x + \frac{2x^3}{3} - 2x^2 + 4x$$

$$= x(\frac{5x^2}{3} - 4x + 6) > 0 \ . \quad \text{(Discriminant} < 0)$$

$$\frac{F''(\frac{1}{x})}{x^4} + \frac{2F'(\frac{1}{x})}{x^3} = \frac{1}{x^6} - \frac{2}{x^5} + \frac{2}{x^4} + \frac{2}{3x^6} - \frac{2}{x^5} + \frac{4}{x^4}$$

$$= \frac{1}{x^6}(\frac{5}{3} - 4x + 6x^2) > 0 \ .$$

If the graph of $x \longrightarrow xF(x)$ is flexed upward (or downward), then the graph of $x \longrightarrow F(\frac{1}{x})$ is flexed upward (or downward).

20. If the graph of f is flexed upward and $F(a) = F(b) = F(c)$ where $a < b < c$, show that $F(x)$ is constant in (a,c) .

If f were not constant, then since the curve lies nowhere above its chord we have $f(u) < f(a)$. But $(b,f(b))$ then lies above both the chords from $(a,f(a))$ to $(u,f(u))$ and from $(u,f(u))$ to $(b,f(b))$, contradicting upward flexure.

21. Given twice differentiable functions f and g whose graphs have upward flexure, with f increasing, show that the graph of fg also has upward flexure.

$f''(x) \geq 0$ for all x ; $f'(x) \geq 0$ for all x .

$g''(x) \geq 0$ for all x .

$$\begin{aligned}[fg(x)]' &= f'g(x) \cdot g'(x) \\ [fg(x)]'' &= f'g(x) \cdot g''(x) + g'(x) \cdot f''g(x) \cdot g'(x) \\ &= f'g(x) \cdot g''(x) + (g'(x))^2 \cdot f''g(x) \\ &> 0 \ .\end{aligned}$$

Thus the graph of fg is flexed upward.

22. (a) Let a , b , c be three points in I such that $a < b < c$, and suppose that the graph of f is flexed upward in I . Show that

$$f(b) \leq \frac{c - b}{c - a} f(a) + \frac{b - a}{c - a} f(c) .$$

(Hint: use the result of Number 14).
Hence, show that

$$f(a) \geq \frac{c - a}{c - b} f(b) - \frac{b - a}{c - b} f(c) ,$$

$$f(c) \geq \frac{c - a}{b - a} f(b) - \frac{c - b}{b - a} f(a) .$$

Similarly to 14(a) (this is less complicated as we are on the number line here rather than the plane), $b = \theta a + (1 - \theta)c$, for some $\theta : 0 < \theta < 1$. Then

$$\theta \cdot (a - c) = b - c$$

or $$\theta = \frac{c - b}{c - a} , (1 - \theta) = \frac{b - a}{c - a} .$$

and $$b = \frac{c - b}{c - a} a + \frac{b - a}{c - a} c .$$

Now from 14(b), we have that

$$f(b) \leq \frac{c - b}{c - a} f(a) + \frac{b - a}{c - a} f(a) ,$$

for the graph of f flexed upwards. Multiplying both sides by $\frac{c - a}{c - b}$, we get $f(a) > f(b) \cdot \frac{c - a}{c - b} - f(c) \cdot \frac{b - a}{c - b}$. Similarly $f(c) \geq f(b) \cdot \frac{c - a}{b - a} - f(a) \cdot \frac{c - b}{b - a}$.

∧ (b) If the graph of F is flexed upward is a closed interval, show that F is bounded in the interval.

It is an immediate consequence of the flexure of f that it is bounded above. Given a closed interval $[\alpha,\beta]$ in I , let the chord connecting $(\alpha,f(\alpha))$, $(\beta,f(\beta))$ be described by the linear function g . Then $g(x) \leq \max\{f(\alpha),f(\beta)\}$ on $[\alpha,\beta]$, and by the upward flexure of f , $f(x) \leq g(x) \leq \max\{f(\alpha),f(\beta)\}$.

To find lower bounds for f , we use part (a). Take any point γ in $[\alpha,\beta]$ which is not an endpoint, so that $\alpha < \gamma < \beta$. For $\alpha \leq x < \gamma$, we have $x < \gamma < \beta$ and

$$f(x) \geq \frac{\beta - x}{\beta - \gamma} f(\gamma) - \frac{\gamma - x}{\beta - \gamma} f(\beta) \qquad \text{by (a)},$$

$$\geq -\left(\frac{\beta - x}{\beta - \gamma} |f(\gamma)| - \frac{\gamma - x}{\beta - \gamma} |f(\beta)|\right)$$

$$\geq -\left(\frac{\beta - \alpha}{\beta - \gamma} \ |f(\gamma)|\right) .$$

Similarly, for $\gamma < x \leq \beta$, we have $\alpha < \gamma < x$ and using (a) again,

$$f(x) \geq -\left(\frac{\beta - \alpha}{\gamma - \alpha} \ |f(\gamma)|\right) .$$

(c) Show by a counter example that the result in (a) is not valid for an open interval.

The graph of $f : x \longrightarrow \frac{1}{x}$ is flexed upward on $(0,x_0)$, $x_0 > 0$, but is unbounded. However, f is always bounded below on a finite open interval as well as finite closed interval, as can be seen from the proof of (b).

23. (a) If the graph of f is flexed upward in an open interval, show that f is continuous in the interval.

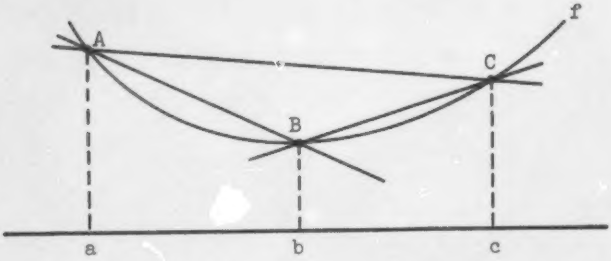
Let a be some point in an open interval I We wish to show f is continuous at a , i.e.,

$$\lim_{x \to a} f(x) = f(a) .$$

a is not an endpoint of I and thus there exists some d such that $[a - d, a + d]$ is contained in I . We first note that as a consequence of 22(a),

$$\frac{f(b) - f(a)}{b - a} \leq \frac{f(c) - f(a)}{c - a} \leq \frac{f(c) - f(b)}{c - b}$$

for a , b , c in an interval I and $a < b < c$. This is a statement about the slopes of the chords connecting $(a,f(a))$, $(b,f(b))$ and $(c,f(c))$ on the graph of f , illustrated in the figure below. The proof is contained in the solution to Number 11.



$$\text{slope } \overleftrightarrow{AB} \leq \text{slope } \overleftrightarrow{AC} \leq \text{slope } \overleftrightarrow{BC}$$

Now if $a < x < a + d$, using these inequalities we have

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(a + d) - f(a)}{d}$$

and using $a - d < a < x$

$$\frac{f(a) - f(a - d)}{d} < \frac{f(x) - f(a)}{x - a}$$

or

$$\frac{f(a) - f(a - d)}{d} \leq \frac{f(x) - f(a)}{x - a} \leq \frac{f(a + d) - f(a)}{x - a} .$$

The same inequality follows if $a - d < x < a$.

So $\left|\frac{f(x) - f(a)}{x - a}\right| \leq k$, k constant, in the deleted 2d-neighborhood of a .

Then $\quad -k|x - a| \leq |f(x) - f(a)| \leq k|x - a|$,
and by the Squeeze Theorem, as $|x - a| \sim 0$, $|f(x) - f(a)| \sim 0$
and $\lim\limits_{x \sim a} f(x) = f(a)$.

(b) Show by a counter-example that the result in (a) is not valid for a closed interval.

The graph of

$$f(x) = \begin{cases} 1 , & x = 0 \\ x , & 0 < x \leq 1 \end{cases}$$

is flexed upward but is not continuous at $x = 0$ i.e., it is not continuous on $[0,1]$.

A24. **If the graph of f is flexed upward in an interval, then f possesses left and right-sided derivatives at each interior point of the interval. (See Exercises 4-2a, No. 9)**

We are given a function f, convex on an interval I, its graph flexed upward. We wish to show that

$$\lim_{x \to \alpha^+} \frac{f(x) - f(\alpha)}{x - \alpha}$$

and

$$\lim_{x \to \alpha^-} \frac{f(x) - f(\alpha)}{x - \alpha}$$

exist, where α is an interior point of I.

We again use the fact that

$$\frac{f(x) - f(\alpha)}{x - \alpha} = g(x)$$

is an increasing function on I.

Since α is in the interior of I, there exist points a and b in I such that $a < \alpha < b$. For x such that $x \neq \alpha$ and $a \leq x \leq b$, we have

$$g(a) \leq g(x) \leq g(b) .$$

On $[a,\alpha)$ and $(\alpha,b]$, g is a bounded monotone function. We wish to show that this implies the existence of left and right-hand limits of g at α, i.e.,

$$g_-(\alpha) = \lim_{x \to \alpha^-} \frac{f(x) - f(\alpha)}{x - \alpha}$$

and

$$g_+(\alpha) = \lim_{x \to \alpha^+} \frac{f(x) - f(\alpha)}{x - \alpha} .$$

These limits are just the left and right-sided derivatives of f at α respectively. We now prove their existence.

We show g bounded and monotone on $[a,\alpha)$ implies a limit at α. The argument for g on $(\alpha,b]$ is very similar. To prove our assertion, we use the concept of least upper bound (l.u.b.) discussed in Appendix 1-5. We claim that

$$\lim_{x \to \alpha} g(x) = \text{l.u.b} \{g(x), x \text{ in } [a,\alpha)\} .$$

Now the least upper bound of the set $g(x)$, x in $[a,\alpha)$, is either the limit of $g(\alpha)$ or $x \sim \alpha$, since g is weakly increasing, or it is a member of the set $g(x)$. If it is a member of the set, say $g(\overline{x})$, then g must be constant or $[\overline{x},\alpha)$, and the limit of $g(x) = c$ at α is c . So

$$\lim_{x \sim \alpha^-} g(\alpha) = \text{l.u.b } \{g(x), x \text{ in } [a,\alpha)\} .$$

Similarly,

$$\lim_{x \sim \alpha^+} g(x) = \text{l.u.b } \{g(x), x \text{ in } (\alpha,b]\} .$$

TC5-6. Constrained Extreme Value Problems.

Extremum problems with constraints involve two or more variables and or more functions of these variables. One of the functions is to be maximized or minimized: the other(s) to be held fixed, thus imposing restrictions or constraint on the variables.

The work in this section presupposes familiarity with implicitly defined functions and their derivatives (Sections 4-8, A5).

Examples have been selected to illustrate techniques that are generally applicable as well as methods appropriate for only certain types of problems. Obviously, there are many ways to attack an extremum problem with constraints (e.g., Exercises 5-6b and 5-4c).

We may picture a line as the intersection of two planes; a curve in a plane as the intersection of a surface and a plane; and a curve in space as the intersection of two surfaces.

If the domain D of the function given by $y = f(x)$ is an interval, then the ordered pairs (x,y) represent points in 2-space over the interval D . Similarly, if $z = f(x,y)$ defines a function with domain D , a region in the xy plane, then the triples (x,y,z) represent points in 3-space over the region D . Consequently, the relation $z = f(x,y)$ may, in general, be represented by a surface.

We may describe this surface in a way which does not require a 3-dimensional sketch. First, we observe that for any value of z , say z_0 , the relation $z_0 = f(x,y)$ describes a curve which is the intersection of the surface $z = f(x,y)$ and the plane $z = z_0$. Then, we obtain a set of such curves by taking various fixed values of z . These curves, called level curves or contour lines, provide a description of the surface.

For the circular paraboloid given by $z = f(x,y) = x^2 + y^2$, level curves corresponding to $z = \frac{125}{18}$, $\frac{125}{2}$, and $\frac{657}{16}$ are shown in Figure 5-6a.

Now we consider a relation $g(x,y) = 0$ in which the z term is absent. We note that the equation $g(x,y) = 0$ composes no restriction on z and hence z takes on all real values (in a 3-dimensional frame of reference). The relation $g(x,y) = 0$ is represented by a cylindrical surface. In Example 5-6a, the relation $g(x,y) = 4y^2 - 16y - 12x - 75 = 0$ is seen as a surface called a parabolic cylinder. The name derives from the fact that for each real value of z, say z_0, the plane $z = z_0$ intersects the cylindrical surface in a parabola.

The parabolic cylinder

$$g(x,y) = 4y^2 - 16y - 12x - 75 = 0$$

meets the surface

$$f(x,y) = x^2 + y^2 = z$$

along a curve in space. In the extreme value problem we locate the high and low points of this curve: The three possible extrema are the points $(-\frac{5}{6}, -\frac{5}{2}, \frac{125}{18})$, $(-\frac{15}{2}, \frac{5}{2}, \frac{125}{2})$, and $(-\frac{9}{4}, 6, \frac{657}{16})$, respectively. Clearly, the low point is $(-\frac{5}{6}, -\frac{5}{2}, \frac{125}{18})$ where $z = \frac{125}{18}$.

Solutions Exercises 5-6

1. (a) For a given volume V find the dimensions of a cylindrical tin can with smallest surface area.

$$V = \pi r^2 h \qquad \frac{dv}{dr} = 2\pi rh + \pi r^2 \frac{dh}{dr} = 0$$

so that $\frac{dh}{dr} = \frac{-2h}{r}$

Surface area, $s = 2\pi rh + 2\pi r^2$

$$\frac{ds}{dr} = 2\pi h + 2\pi r \frac{dh}{dr} + 4\pi r$$
$$= 2\pi[2r - h]$$

$\frac{ds}{dr}$ is zero if $h = 2r$

$\frac{d^2s}{dr^2} = 2\pi(2 - \frac{dh}{dr})$ is 12π if $h = 2r$.

Hence by Theorem 5-5a the surface area is minimum if $h = 2r$, where

$$r = (\frac{v}{2\pi})^{1/3} \text{ and } h = 2(\frac{v}{2\pi})^{1/3}.$$

(b) If the cost of the sides, top, and bottom of the can is a cents per square inch, and if the cost of the bead joining the top and bottom to the side is b cents per linear inch, find the most economical dimensions of the can for a given volume V.

Let cost = C.

$$C = 2a\pi(rh + r^2) + 4b\pi r$$

$$D_rC = 2\pi a(r \frac{dh}{dr} + h + 2r) + 4b\pi$$

$$= 2\pi(-ah + 2ar + 2b) \text{ , since}$$

$\frac{dh}{dr} = \frac{-2h}{r}$, from (a).

D_rC is zero if $h = 2r + \frac{2b}{a}$,

where r is the positive solution of the equation

$$V = 2\pi(r^3 + \frac{b}{a}r^2) .$$

To show that these are most economical dimensions we apply Theorem 5-5a.

$$D_r[2\pi(-ah + 2ar + 2b)] = 2\pi(-a \frac{dh}{dr} + 2a) = 2\pi[6a + \frac{4b}{r}] > 0$$

for $h = 2r + \frac{2b}{a}$.

2. (a) A cylindrical sheet-iron tank without top is to have volume V. Let h be the height of the tank, and r, the radius of the base. The side of the tank is to be constructed from stock costing P dollars per square foot and the base from stock costing Q dollars per square foot. Find the radius and height of the tank for which the cost of material in the tank is minimized.

$$V = \pi r^2 h \qquad h = \frac{v}{\pi r^2}$$

$$\text{Cost} = C = P(2\pi rh) + Q\pi r^2$$

$$= \frac{2Pv}{r} + Q\pi r^2 = f(r)$$

$$f'(r) = \frac{2(\pi Q r^3 - Pv)}{r^2} \text{ is zero if } r = (\frac{Pv}{\pi Q})^{1/3}$$

$$f''(r) = \frac{4Pv}{r^3} + 2\pi Q > 0 \text{ for all } r > 0 .$$

Hence by Theorem 5-5a, the radius and height for which cost is minimized is

$$r = (\frac{Pv}{\pi Q})^{1/3} , h = \frac{Q}{P}(\frac{Pv}{\pi Q})^{1/3} = \frac{Q}{P} r .$$

(b) More realistically, suppose that the base has to be cut from a square of side $2r$. Find the dimensions yielding minimum cost including the cost of material trimmed away.

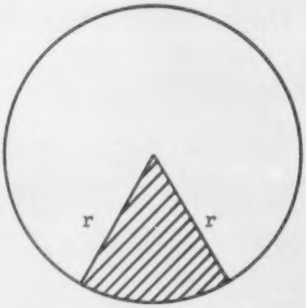
$$\text{Cost} = C = \frac{2Pv}{r} + 4Qr^2 = q(r)$$

$$g'(r) = \frac{8r^3Q - 2Pv}{r^2}$$

Minimum cost if

$$r = \left(\frac{VP}{4Q}\right)^{1/3} , \quad h = \frac{4Q}{\pi P}\left(\frac{VP}{4Q}\right)^{1/3} = \frac{4Q}{\pi P} r \ .$$

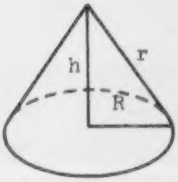
3. Find the radius and height of the cone of greatest volume that can be made from a circular sheet of radius r by cutting a wedge from the center and bending the remaining portion to form a cone.



Let R denote the radius of base of cone and h , the height. Since $r^2 = h^2 + R^2$ is a constant,

$$0 = 2h + 2R\frac{dR}{dh} ,$$

whence $\frac{dR}{dh} = \frac{-h}{R}$



$v = \frac{1}{3}\pi R^2 h$ so that

$$\frac{dv}{dh} = \frac{2}{3}\pi Rh\frac{dR}{dh} + \frac{1}{3}\pi R^2 \ .$$

$\frac{dv}{dh} = \frac{\pi}{3}(R^2 - 2h^2)$ is zero if $R = \sqrt{2}\,h$.

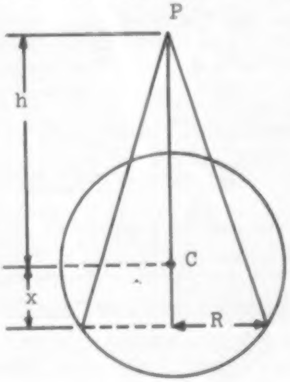
$\frac{d^2v}{dh^2} = \frac{\pi}{3}(2R\frac{dR}{dh} - 4h) = -2h\pi < 0$ for $h > 0$.

Maximum volume is obtained if

$$h = \frac{\sqrt{3}}{3}r \quad \text{and} \quad R = \frac{\sqrt{6}}{3}r .$$

4. A point P is at a distance h above the center C of a sphere of radius r , where h > r . A cone is constructed having P for vertex, and for base the circle formed by cutting the sphere with a plane perpendicular to PC . In order to have the volume of the cone as great as possible, should this plane be above or below C ? How far?

Suppose the plane is above C . If we move the plane toward C the height of the cone and the area of the base will each increase so that no maximum volume is reached.



Therefore the place is at C or below C . Let R denote the radius of the cone and h + x the altitude, where x denotes the distance from the center C to the plane of the base of the cone. Then the volume,

$$v = \frac{\pi}{3} R^2(h + x) \text{ , and}$$

$$\frac{dv}{dx} = \frac{\pi}{3}[2R(h + x)\frac{dR}{dx} + R^2] .$$

Since $r^2 = R^2 + x^2$,

$$0 = 2R \frac{dR}{dx} + 2x \quad \text{and} \quad \frac{dR}{dx} = \frac{-x}{R} .$$

It follows that

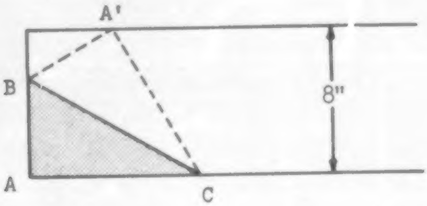
$$\frac{dv}{dx} = \frac{\pi}{3}[-2\pi h - 2x^2 + R^2]$$

$$= \frac{\pi}{3}(-3x^2 - 2xh + r^2)$$

is zero if $x = \frac{-h + \sqrt{h^2 + 3r^2}}{3}$.

Since $\frac{d^2v}{dx^2} = \frac{-2\pi}{3}(3x + 1)$, by Theorem 5-5a, the volume of the cone is maximum if $x = \frac{-h + \sqrt{h^2 + 3r^2}}{3}$, the distance from C to the plane, where the plane is below C .

5. A long strip of paper 8 inches wide is cut off square at one end. A corner A of this end is then folded over to the opposite side at A' , thus forming triangle ABC . Find the area of the smallest triangle that can be formed in this way.



Let $AB = BA' = x$

Then $BD = 8 - x$

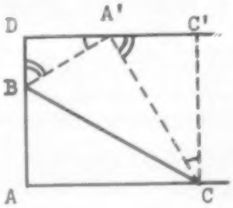
and $DA' = 4\sqrt{x - 4}$ by the Pythagorean Theorem

$$AC = A'C = \frac{2x}{\sqrt{x - 4}}$$ by similar triangles DA'B and C'CA' .

Area of Triangle $ABC = \dfrac{x^2}{\sqrt{x - 4}} = f(x)$.

$f'(x) = \dfrac{3x^2 - 16x}{2(x - 4)^{3/2}}$ is zero if $x = \dfrac{16}{3}$

$f''(\frac{16}{3}) > 0$ and hence $f(\frac{16}{3})$ is minimum value. The area of the smallest triangle is $f(\frac{16}{3}) = \dfrac{128\sqrt{3}}{9}$.



6. (a) Let the general equation of a straight line L be given in the form

$$ax + by = c .$$

Find the point Q on L for which the distance to a given point P not on L is a minimum. Prove that the line joining P to Q is perpendicular to L .

Let $P = (u,v)$ be the fixed point and $Q = (x,y)$ be the point on L . If s is the distance from P to Q then

$$z = s^2 = (x - u)^2 + (y - v)^2$$

and this quantity is to be minimized. At a minimum we have, on differentiating with respect to x ,

(1) $$z' = 2(x - u) + 2(y - v)y' = 0 .$$

From $ax + by = c$, we have

(2) $$y' = -\frac{a}{b} .$$

Entering (2) in (1) we obtain

$$a(y - v) = b(x - u)$$

and we see that the line PQ is perpendicular to L . Setting

$$a(x - u) + b(y - r) = c - au - bv = w$$

we obtain for the coordinates of Q

$$x = u + \frac{aw}{a^2 + b^2} \text{ and } y = v + \frac{aw}{a^2 + b^2} .$$

(b) On the curve C given by $f(x,y) = 0$ let Q be the point nearest to a point P not on the curve. If Q is not an endpoint of C , and all necessary derivatives exist, prove that the line joining P to Q is perpendicular to C .

Let $P = (u,v)$ denote the given point, $Q = (x,y)$ the point on C and s the distance from P to Q . We have

$$s^2 = (x - u)^2 + (y - v)^2 .$$

If s is a minimum and (x,y) is not an endpoint of C then, on differentiating with respect to x , we have

$$2(x - u) + 2(y - v)y' = 0 .$$

The slope of the line PQ is then

$$\frac{y - v}{x - u} = -\frac{1}{y'} .$$

Since the product of the slope of PQ and the slope y' of C is -1 , we have established perpendicularity.

7. Find the extrema of $x^2 + y^2$ if x and y are subject to the constraint $x^2 - 12x + y^2 - 8y + 51 = 0$. Give a geometric interpretation.

Differentiating $x^2 + y^2$ with respect to x we obtain $2x + 2yy'$.
Let $2x + 2yy' = 0$.

From the constraint condition, we also have

$$2x - 12 + (2y - 8)y' = 0 .$$

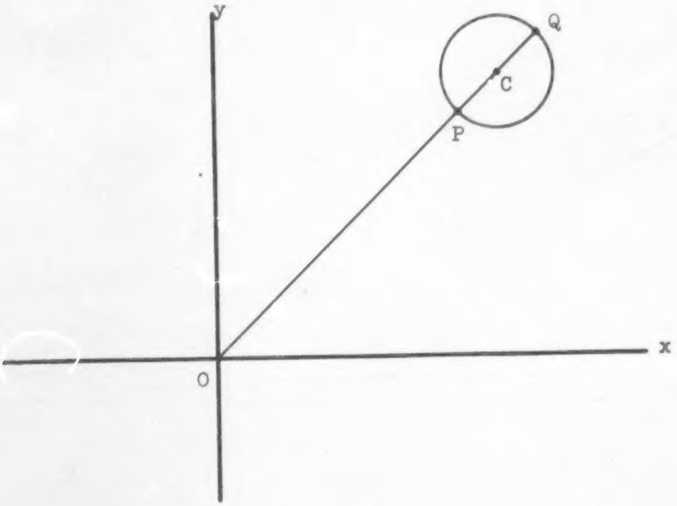
Therefore

$$2x + 2y\left(\frac{12 - 2x}{2y - 8}\right) = 0$$

or
$$y = \frac{2}{3}x .$$

With $y = \frac{2}{3}x$ and $x^2 - 12x + y^2 - 8y + 51 = 0$, we obtain the following maximum and minimum points $(6 + \frac{3\sqrt{13}}{13}, 4 + \frac{2\sqrt{13}}{13})$, $(6 - \frac{3\sqrt{13}}{13}, 4 - \frac{2\sqrt{13}}{13})$. Thus the extreme values of $x^2 + y^2$ are $x^2 + \frac{4x^2}{9} = (2\sqrt{13} \pm 1)^2$. Geometrically, we note that we are maximizing the square of the distance between the origin and the curve $x^2 - 12x + y^2 - 8y + 51 = 0$.

Alternatively, we first note that the constraint condition is that x and y lie on the circle $(x - 6)^2 + (y - 4)^2 = 1$, (center (6,4), radius 1). Since $\sqrt{x^2 + y^2}$ is the distance between the point (x,y) and the origin, we have to find the closest and farthest points of the given circle to the origin. These are the points P and Q on the line connecting the center to the origin



Thus, the extreme values of $x^2 + y^2$ are

$$[OC \pm 1]^2 \text{ or } (\sqrt{6^2 + 4^2} \pm 1)^2 = (2\sqrt{13} \pm 1)^2 .$$

8. If $x_i \geq 0$, $i = 1 , 2 , \ldots , n$
and $x_1 + x_2 + \ldots + x_n = s$ (constant),
find the maximum value of the product

$$x_1 \cdot x_2 \ldots x_n$$

(assuming it exists).

Assume the maximizing values for x_i , $i \geq 3$ are given by

$$x_i = a_i ,$$

then we have to determine x_1 and x_2 such that

$$x_1 + x_2 = s - (a_3 + a_4 + \ldots + a_n)$$

and x_1x_2 is a maximum.

This latter problem has been solved previously (Section 1-1) and, consequently, $x_1 = x_2$. By the same argument we conclude that $x_1 = x_3$, $x_1 = x_4$, etc.; hence, $x_1 = x_2 = \ldots = x_n = \frac{s}{n}$.

TC5-7. Tangent and Normal Lines.

The postponement to this section of the discussion of tangent lines was deliberate. With the Law of the Mean as a basis, we are now ready to exploit properties of tangent lines and justify the use of their equations to approximate functional values.

Miscellaneous Exercises, Numbers 1-3, are pertinent to this section and may be assigned at any time: the discussion of Number 3 includes general consideration of the number of tangent lines that can be drawn from a given point to a given conic section.

Solutions Exercises 5-7

1. Show that the number of tangent lines that can be drawn from the point (h,k) to the curve $y = x^2$ is two, one, or zero, according to whether k is less than h^2 , equal to h^2 , or greater than h^2 , respectively.

If $y = x^2$, $y' = 2x$.

Let (x_0,y_0) be a point of tangency on $y = x^2$. Then an equation of the line through the point (x_0,y_0) with slope $2x_0$ is

(1) $$y = 2x_0(x - x_0) + y_0 .$$

The line joining the point (x_0,y_0) to (h,k) is given by

(2) $$y = \frac{y_0 - k}{x_0 - h}(x - x_0) + y_0 .$$

Equations (1) and (2) describe the same line if the slopes are equal; that is, if

$$2x_0 = \frac{y_0 - k}{x_0 - h} = \frac{x_0^2 - k}{x_0 - h}$$

or $x_0^2 - 2x_0h + k = 0$. This equation has solutions

$$x_0 = h \pm \sqrt{h^2 - k} .$$

If $k < h^2$, there are two solutions and hence two points of tangency, i.e., two tangent lines. If $k = h^2$, there is exactly one solution, one tangent line. If $k > h^2$, there are no solutions and hence no tangent line as specified.

Interpreted geometrically, this theorem states that

(i) no tangent line to $y = x^2$ passes through the convex region $y > x^2$;

(ii) through any point on $y = x^2$, only one tangent line to $y = x^2$ can be drawn, namely, the tangent through that point;

(iii) through any point "exterior" to the parabola, i.e., any point in the region $y < x^2$, two tangent lines to the parabola may be drawn.

2. Find equations for the tangent and normal lines to the graphs of the following functions at the given points.

(a) $f : x \to x\ \mathrm{si}$[illegible] , $x = 0$, $x = \frac{\pi}{2}$.

$f'(x) = x \cos x + \sin x$

Tangent line at $x = 0$: $y = 0$.

Normal line at $x = 0$: $x = 0$.

Tangent line at $x = \frac{\pi}{2}$: $y = x$.

Normal line at $x = \frac{\pi}{2}$: $y = -x + \pi$.

(b) $f : x \to \arcsin(\frac{1}{x})$, $x = 2$.

$$f'(x) = \frac{-1}{x\sqrt{x^2 - 1}} .$$

$f(2) = \frac{\pi}{6}$, $f'(2) = \frac{-\sqrt{3}}{6}$.

Tangent line at $x = 2$: $y = \frac{-\sqrt{3}}{6}(x - 2) + \frac{\pi}{6}$.

Normal line at $x = 2$: $y = 2\sqrt{3}\,(x - 2) + \frac{\pi}{6}$.

(c) $f : x \to \dfrac{x^3}{1 + x^2}$, $x = -1$, $x = 0$.

$$f'(x) = \frac{x^2(3 + x^2)}{(1 + x^2)^3} .$$

Tangent line at $x = -1$: $y = x + \frac{1}{2}$.

Normal line at $x = -1$: $y = -x - \frac{3}{2}$.

Tangent line at $x = 0$: $y = 0$.

Normal line at $x = 0$: $x = 0$.

(d) $f : x \to \dfrac{x}{x^2 - 1}$, $x = x_0$ where $|x_0| \neq 1$.

$$y' = f'(x) = \frac{-(x^2 + 1)}{(x^2 - 1)^2} .$$

Tangent line at $x = x_0$, where $|x_0| \neq 1$:

$$y = \frac{-(x_0^2 + 1)}{(x_0^2 - 1)^2}(x - x_0) + \frac{x_0}{x_0^2 - 1} .$$

Normal line at $x = x_0$, where $|x_0| \neq 1$:

$$y = \frac{(x_0^2 - 1)^2}{x_0^2 + 1}(x - x_0) + \frac{x_0}{x_0^2 - 1} .$$

3. (a) For the ellipse

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$$

and the hyperbola

$$\frac{x^2}{p^2} - \frac{y^2}{q^2} = 1 ,$$

obtain equations of the lines tangent at (a,b) on each curve in a symmetrical form like that of Example 5-7b for the circle.

For the ellipse $\dfrac{x^2}{p^2} + \dfrac{y^2}{q^2} = 1$, the tangent line at (a,b) has equation

$$\frac{ax}{p^2} + \frac{by}{q^2} = \frac{a^2}{p^2} + \frac{b^2}{q^2} = 1 .$$

For the hyperbola, the line tangent at (a,b) is given by

$$\frac{ax}{p^2} - \frac{by}{q^2} = \frac{a^2}{p^2} - \frac{b^2}{q^2} = 1 .$$

(b) For the same two curves, obtain equations for the normal lines at (a,b) on each curve in an analogous form.

For the ellipse, the normal line at (a,b) is given by

$$\frac{bx}{q^2} - \frac{ay}{p^2} = ab\left(\frac{1}{q^2} - \frac{1}{p^2}\right) .$$

For the hyperbola, the normal line at (a,b) is given by

$$\frac{bx}{q^2} + \frac{ay}{p^2} = ab\left(\frac{1}{q^2} + \frac{1}{p^2}\right) .$$

4. Prove that the line tangent to the circle of Example 5-7b at (a,b) on the circle, meets the circle at no other point.

If $ax + by = r^2$, then $x = \frac{r^2 - by}{a}$. Substituting this value in $x^2 + y^2 = r^2$, we have

$$\left(\frac{r^2 - by}{a}\right)^2 + y^2 = r^2$$

or $r^2y^2 - 2br^2y + r^2b^2 = 0$, since $a^2 + b^2 = r^2$. This equation has but one solution, $y = b$, since the discriminant is zero. Therefore, the tangent line $ax + by = r^2$ meets the circle in only one point (a,b).

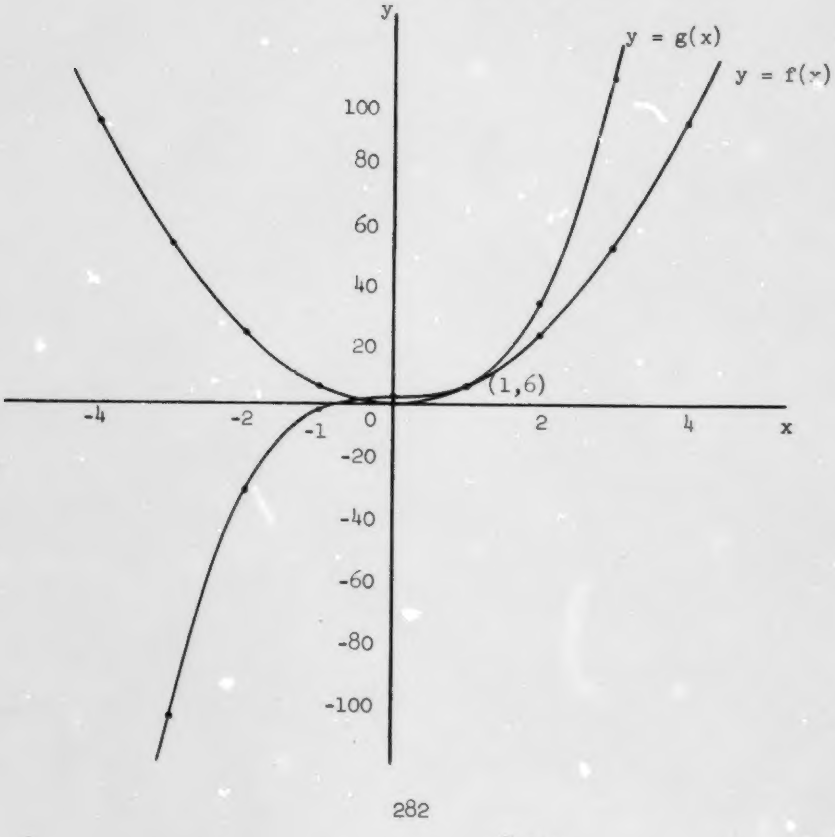
5. Show that the graph of the functions f and g where $f : x \to 6x^2$ and $g : x \to 4x^3 + 2$ have a common tangent line at the point $(1,6)$. Sketch the graph.

We need to show that $f'(x) = g'(x)$ and $f(x) = g(x)$ for $x = 1$.

$$f'(x) = 12x \; ; \; g'(x) = 12x^2 .$$

So

$$f(1) = g(1) = 6 \; ; \; f'(1) = g'(1) = 12 .$$



6. If $f : x \to ax^2 + bx + c \quad (a \neq 0)$, show that the tangent line to the graph of f at the point $(p, f(p))$ is parallel to the chord joining the points $(m, f(m))$ and $(n, f(n))$ only if $p = \frac{m+n}{2}$.

The tangent line at $(p, f(p))$ has slope $f'(p)$; the chord joining $(m, f(m))$ and $(n, f(n))$ has slope $\frac{f(m) - f(n)}{m - n}$. The chord and the tangent will be parallel only if their slopes are equal, i.e., only if

$$f'(p) = \frac{f(m) - f(n)}{m - n},$$

or, as $f'(p) = 2ap + b$,

$$\begin{aligned} 2ap + b &= \frac{f(m) - f(n)}{m - n} \\ &= \frac{a(m^2 - n^2) + b(m - n)}{m - n} \\ &= a(m + n) + b, \end{aligned}$$

if we assume that $m \neq n$. So we have $2ap + b = a(m + n) + b$ or $p = \frac{m+n}{2}$.

7. Given the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and an arbitrary point P on the curve but not on either axis. Prove that if the normal at P to the ellipse passes through the origin, then the ellipse is a circle.

$b^2x^2 + a^2y^2 = a^2b^2$.

The normal at $P(x_0, y_0)$ has equation

$$\frac{x_0 y}{a^2} - \frac{y_0 x}{b^2} = x_0 y_0 \left(\frac{1}{a^2} - \frac{1}{b^2}\right).$$

If the normal at P passes through the origin then $x = 0$, $y = 0$ satisfies the above equation so that

$$\frac{1}{a^2} - \frac{1}{b^2} = 0$$

or

$$b^2 = a^2,$$

whence the original equation is

$$x^2 + y^2 = a^2,$$

the equation of a circle.

8. Comment on the footnote in Section 1-1, page 6.

We are interested in the value of f, twice differentiable, at some point $x = a$. However, we only know the value of f at some point $\bar{a}$, close to a, say $\bar{a} < a$. We estimate the error in approximating $f(a)$ by $f(\bar{a})$ by use of the Law of the Mean. Applying the Law of the Mean twice, we have

$$f(\bar{a}) - f(a) = (\bar{a} - a)f'(u) , \qquad \bar{a} < u < a$$

and

$$f'(u) = f'(a) = (u - a)f''(v) , \qquad u < v < a .$$

Then

$$|f(\bar{a}) - f(a)| = |(\bar{a} - a)f'(a) + (\bar{a} - a)(u - a)f''(v)| .$$

Since $|u - a| < |\bar{a} - a|$,

$$\begin{aligned} |f(\bar{a}) - f(a)| &\le |(\bar{a} - a)f'(a)| + |(\bar{a} - a)(u - a)f''(v)| \\ &< (\bar{a} - a)|f'(a)| + (\bar{a} - a)^2|f''(v)| . \end{aligned}$$

Now, if $f(a)$ is at or near a maximum (peak), $|f'(a)|$ is very small and $|f(\bar{a}) - f(a)|$ is nearly second order in $|\bar{a} - a|$. In that case, for $\bar{a}$ close to a, the error in approximating $f(a)$ by $f(\bar{a})$ is small.

As an application, suppose we are to obtain estimates for $\sin 89^\circ$ and $\sin 5^\circ$. Suppose the angles are measured to within $\pm .5^\circ$, so that the experimentally determined value obtained for $\sin 89^\circ$ lies between $\sin 88.5^\circ$ and $\sin 89.5^\circ$. Then the error in estimating $\sin 89^\circ$ and $\sin 5^\circ$ from the given measurements is much less for the former, since $f'(89^\circ) \approx 0$ and $f'(5^\circ) \approx 1$. To illustrate numerically,

$$\begin{aligned} \sin 89.5^\circ &= .99996 , \\ \sin 88.5^\circ &= .99966 , \\ \sin 5.5^\circ &= .09585 , \\ \cos 4.5^\circ &= .07846 , \\ \sin 89.5^\circ - \sin 88.5^\circ &= .00030 , \\ \sin 5.5^\circ - \sin 4.5^\circ &= .01739 . \end{aligned}$$

The error in estimating $\sin 89^\circ$ is less than $\frac{1}{50}$-th the error in estimating $\sin 5^\circ$.

9. (a) Estimate the error of approximation to $y = \sin x$ by the tangent at $x = 0$.

From the Law of the Mean we have

$$\sin x - \sin 0 = x \cos u ,$$

where u lies between 0 and x. Similarly,

$$\cos u - \cos 0 = \cos u - 1 = -u \sin v$$

where v lies between 0 and u. Combining the results we have

$$\sin x = x \cos u = x(1 - u \sin v)$$

or

$$x - \sin x = xu \sin v .$$

Knowing that $0 < |v| < |u| < |x|$ and $|\sin v| < 1$, we obtain an estimate of the type in the text, namely,

$$|x - \sin x| \leq x^2 .$$

(b) Show that the error is at least third order in $|x|$, i.e., that

$$|x - \sin x| \leq c|x|^3 ,$$

where c is constant.

We carry the argument of (a) one step further. We have, as in the first line above,

$$\sin v = v \cos w$$

where w lies between 0 and v. Hence,

$$x - \sin x = xuv \cos w ,$$

whence

$$|x - \sin x| \leq |x|^3 .$$

Thus x is an excellent approximation to $\sin x$ for small angles.

10. Compare the methods of this section with the methods of Section 5-3 and Exercises 5-3, Numbers 15 and 16.

The interesting question is the relative precision of the two methods of estimation. The error in approximation to $f(x)$ for x in (p,q) is bounded above by $e = (x - p)(q - p)M_2$ where M_2 is an upper bound for $|f''(x)|$ on (p,q). The error in the tangent approximation is bounded by $e^* = (x - a)^2 M_2^*$ where M_2^* is an upper bound for $|f''(x)|$ in the interval between a and x.

In Example 5-3b, we had $x = 10$, $p = 9$, $q = 16$, and we used $|f''(x)| = \frac{1}{4x^{3/2}} < \frac{1}{4 \cdot 9^{3/2}} \le \frac{1}{108}$ in (p,q) to obtain $e < \frac{7}{108}$. In the tangent approximation, we use $a = 9$, and the same bound on the derivative to obtain $e^* < \frac{1}{108}$. Thus, independently of which of the approximations to $\sqrt{10}$ is better, the tangent approximation gives us surer information about $\sqrt{10}$ in this case.

In Exercises 5-3, Number 15, observe for $f(x) = \sqrt[3]{x}$ that $|f''(x)|$ is a decreasing function of x for $x > 1$. In approximation to $\sqrt[3]{1.008}$ by linear interpolation, take $p = 1$, $q = (1.003)^3$, by the tangent method, $p = 1$. Use $M_2 = M_2^* = |f''(1)|$. Then

$$\frac{e}{e^*} = \frac{(1.003)^3 - 1}{.008} < \frac{9.1}{8} < 1.2 ,$$

so we see that the tangent method offers improvement in precision.

A similar analysis applies in Exercises 5-3, Number 16. Here the difficulty in the linear interpolation method is the choice of a point q where $\cos q$ is known and is close enough to $\frac{\pi}{3}$ and suitable for the purposes of precise approximation.

11. Show how to approximate $\sqrt[n]{2^n + 1}$ and estimate the error of approximation.

We shall approximate the graph of $f(x) = x^{1/n}$ by its tangent line at $x = 2^n$. We have

$$f'(x) = \frac{1}{n} \cdot x^{\frac{1}{n} - 1} ,$$

$$f'(2^n) = \frac{1}{n} \cdot (2^n)^{1/n} \cdot (2^n)^{-1} = \frac{1}{n \cdot 2^{n-1}} .$$

The equation of the tangent line through $(2^n,2)$ is

$$y = \frac{1}{n \cdot 2^{n-1}} (x - 2^n) + 2 .$$

If $x = 2^n + 1$,

$$y = \frac{1}{n \cdot 2^{n-1}} + 2 .$$

The absolute error of this value as an approximation for $f(2^n + 1)$ is given by

$$e \le M\left((2^n + 1) - 2^n\right)^2 \le M ,$$

where M is an upper bound for $|f^n(x)|$ on $(2^n, 2^{n+1})$.

$$f''(x) = \left(\frac{1}{n^2} - \frac{1}{n}\right) x^{\frac{1}{n} - 2} .$$

$f''(x)$ is decreasing and positive and so we have the bound

$$M = f''(2^n) = \left(\frac{1}{n^2} - \frac{1}{n}\right) \cdot \frac{1}{2^{2n-1}} .$$

12. The location of an object at time t on a straight line is given by the law of motion

$$s = 5 \sin 3t - 3 \sin 5t .$$

Once it begins when does the particle first reach a stop? How far is it then from the starting point?

$$s = 5 \sin 3t - 3 \sin 5t .$$

$$\begin{aligned}
\frac{ds}{dt} &= 15 \cos 3t - 15 \cos 5t \\
&= -15(\cos 5t - \cos 3t) \\
&= -15\big(\cos(4t + t) - \cos(4t - t)\big) \\
&= -15(-2 \sin 4t \sin t) \\
&= 30 \sin 4t \sin t \\
&= 0 \quad \text{only if} \quad \sin 4t = 0 \quad \text{or} \quad \sin t = 0 .
\end{aligned}$$

So $\frac{ds}{dt} = 0$ at $t = \frac{\pi}{4}$, the first stop after the particle has begun, i.e., $t > 0$. At $t = \frac{\pi}{4}$,

$$\begin{aligned}
s &= 5 \sin \frac{3\pi}{4} - 3 \sin \frac{5\pi}{4} \\
&= 4\sqrt{2} .
\end{aligned}$$

13. Find an equation of the tangent line to the folium of Descartes

$$x^3 + y^3 - 3axy = 0$$

at the point $(x_0\ y_0)$. Note particularly the situation at the point $(0,0)$.

Let $F(x,y) = x^3 + y^2 - 3axy = 0$.

$$3x^2 + 3y^2y' - 3axy' - 3ay = 0$$

$$y' = \frac{x^2 - ay}{ax - y^2} , \quad ax - y^2 \neq 0 .$$

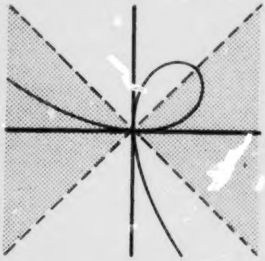
(Note that $ax - y^2 \neq 0$ if $x \neq 0$.) The tangent line to the curve

$F(x,y) = 0$ at (x_0,y_0) is

$$y = \frac{x_0^2 - ay_0}{ax_0 - y_0^2}(x - x_0) + y_0 \quad , \quad x_0 \neq 0 .$$

We observe, moreover, that vertical tangents may occur when $\frac{dx}{dy} = 0$; namely, among the points (x,y) on the graph where $ax - y^2 = 0$. This condition is satisfied at the points $(0,0)$ and $(\sqrt[3]{\frac{4}{a}}, \sqrt[3]{2a})$. At $(0,0)$ the expression for y' is not defined. In fact, the curve crosses itself at $(0,0)$ and does not have a tangent there in the usual sense, but does have two tangents in a sense clarified by parametric representation (Chapter 11).

The situation can be analyzed in the present context in the following way. If a curve has a tangent (non-vertical) at the origin, then its slope is $\lim\limits_{x \sim 0} \frac{y}{x}$. For definiteness take $a > 0$ (for $a < 0$, the curve is the reflection in the origin of $x^3 + y^3 - 3|a|xy = 0$ and the problem is essentially the same). We separate off one of the branches of the curve in a neighborhood of $x = 0$ by restricting $|y| \leq |x|$.



Now, using $|y| \leq |x|$ we see that, for $x \neq 0$,

$$\left|\frac{y}{x}\right| = \left|\frac{x^3 + y^3}{3ax^2}\right| < \frac{2}{3a}\,|x| .$$

Thus, $\lim\limits_{x \sim 0} \frac{y}{x} = 0$ and the curve has a horizontal tangent at $(0,0)$. From the symmetry, we see that it also has a vertical tangent.

It remains to be proved that the conditions $F(x,y) = 0$ and $|y| \leq |x|$ really define y as a function of x . For this, use the techniques of Section 4-8. Namely, for a fixed x , $x = \xi$ where $\xi \neq 0$, define

$$g(y) = F(\xi,y) = \xi^3 + y^3 - 3a\xi y .$$

The derivative

$$g'(y) = 3(y^2 - a\xi)$$

has constant sign for $y < a\sqrt{|\xi|}$, hence $g(y)$ is strongly monotone in $[-a\sqrt{|\xi|}\ , a\sqrt{|\xi|}]$. Now, we have

$$g(a\xi) = \xi^2(\xi[1 + a^3] - 3a^2)$$

and

$$g(-a\xi) = \xi^2(\xi[1 - a^3] + 3a^2)$$

so that g takes opposite signs at these two points for sufficiently small $|\xi|$. We conclude that there is a unique value η for which $F(\xi,\eta) = 0$.

14. Find the equation of the tangent line to the graph of the equation

$$x^2 - x\sqrt{xy} - 2y^2 = 6$$

at the point $(4,1)$.

$$x^2 - x^{3/2} \cdot y^{1/2} - 2y^2 = 6$$

$$2x - \frac{3}{2} x^{1/2} y^{1/2} - \frac{1}{2} x^{3/2} y^{-1/2} y' - 4yy' = 0 .$$

At the point $(4,1)$, the slope is $\frac{5}{8}$. The tangent line at $(4,1)$ is given by

$$8y - 5x + 12 = 0 .$$

TC5-8. Sketching of Graph.

This section provides a fitting conclusion to the chapter: the student is given an opportunity to examine local and global properties of a function and then summarize his findings in a sketch of the graph. The work here draws upon analytic geometry as well as the calculus and unifies both. Given two isolated points (conveniently located) on a curve, we now are in a position to determine whether the points should be connected (e.g., points $(a,0)$ and $(b,0)$ in Miscellaneous Exercises, No. 5(c), (d)). Furthermore, we can isolate intervals where the graph is wild; we can find familiar curves which serve as adequate models for the required graph over certain intervals. The model may be a straight line, as in the case of asymptotes, or graph of the familiar function $x \to x^n$.

The discussion of Miscellaneous Exercises Number 5 in the Teacher's Commentary provides a guide for for sketching graphs of relations defined by rational expressions.

We leave the selection of exercises to you. A wide variety (in terms of level of difficulty, degree of abstraction, depth of insight required) is provided, but this is not intended to suggest that one assignment should comprise the entire range of variation.

Solutions Exercises 5-8

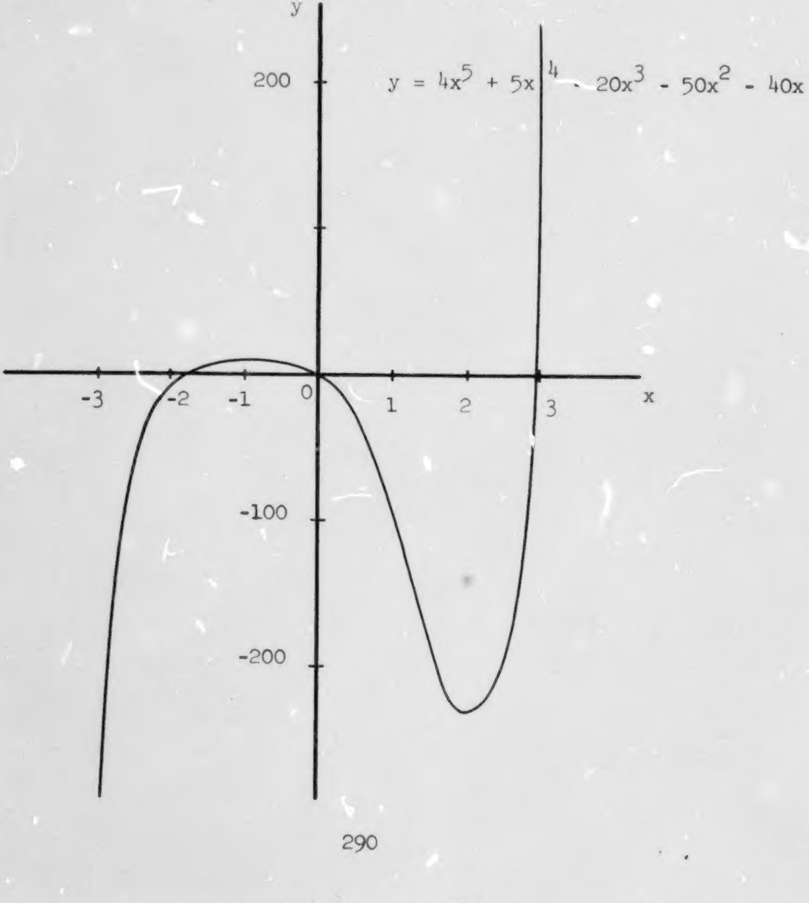
1. Draw the graph of $f : x \rightarrow 4x^5 + 5x^4 - 20x^3 - 50x^2 - 40x$. (See Example 5-8a.)

$$f(x) = 4x^5 + 5x^4 - 20x^3 - 50x^2 - 40x$$

$$f'(x) = 20(x + 1)^3(x - 2)$$

$$f''(x) = 20(x + 1)^2(4x - 5)$$

The graph is flexed downward for $x < \frac{5}{4}$ and flexed upward for $x > \frac{5}{4}$; $f(-1)$ is a local maximum, and $f(2)$ is a local minimum. $f(-1) = 11$, $f(0) = 0$, $f(\frac{5}{4}) = -\frac{454}{5}$, $f(2) = -232$.



2. Locate the point of inflection on the graph of $f : x \to (x + 1)\arctan x$.

$f(x) = (x + 1)\arctan x$.
We have $f''(x) = \dfrac{2(1 - x)}{(1 + x^2)^2}$; since $f''(1) = 0$ and $f''(x)$ changes sign as x increases through the value 1 , it follows that $(1 , \frac{\pi}{2})$ is a point of inflection.

3. Determine equations of the horizontal and vertical asymptotes, if any, of the graph of

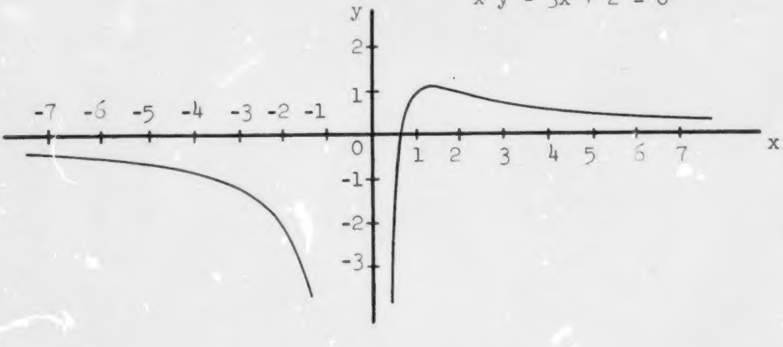
(a) $xy + y = x$.

If $xy + y = x$, then $y = \dfrac{x}{1 + x}$. Now, $\lim\limits_{x \to -1} \dfrac{1}{y} = \lim\limits_{x \to -1} \left(\dfrac{1 + x}{x}\right) = 0$. It follows that the line $x = -1$ is a verticle asymptote. Also $\lim\limits_{x \to \infty} y = \lim\limits_{x \to -\infty} y = 1$, whence the line $y = 1$ is a horizontal asymptote.

(b) $x^2y - 3x + 2 = 0$.

$y = \dfrac{3x - 2}{x^2}$. We have, $\lim\limits_{x \to \infty} y = \lim\limits_{x \to -\infty} y = 0$, whence the line $y = 0$ is a horizontal asymptote. Also $\lim\limits_{x \to 0} \dfrac{1}{y} = \lim\limits_{x \to 0} \dfrac{x^2}{3x - 2} = 0$, whence the line $x = 0$ is a vertical asymptote. (See also Solutions Miscellaneous Exercises, No. 5.)

$x^2y - 3x + 2 = 0$



4. Find horizontal and vertical asymptotes, maxima, minima, and inflection points. Show all tests used to identify each such point and draw the graph of the function.

(a) $f : x \to \frac{x}{x^2 + 1}$ (b) $f : x \to \frac{x^2}{x^2 + 1}$.

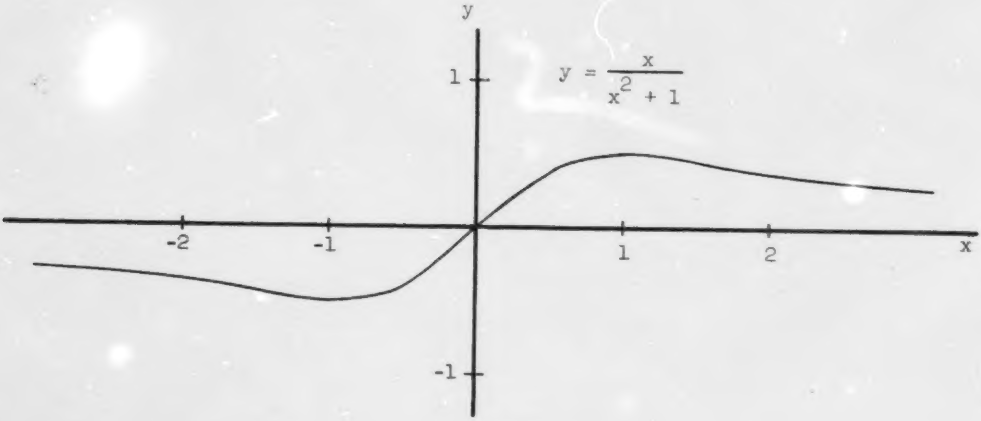
For parts (a) and (b) refer to Solutions Miscellaneous Exercises, Number 5.

(a) $f(x) = \frac{x}{x^2 + 1}$

$f'(x) = \frac{1 - x^2}{(x^2 + 1)^2}$

$f''(x) = \frac{2x(x^2 - 3)}{(x^2 + 1)^3}$

Since $f'(1) = 0$ and $f''(1) < 0$, by Theorem 5-5a, $(1, \frac{1}{2})$ is a maximum point. Similarly, $(-1, -\frac{1}{2})$ is a minimum point. By Definition 5-8, $(\sqrt{3}, \frac{\sqrt{3}}{4})$ and $(-\sqrt{3}, \frac{-\sqrt{3}}{4})$ are points of inflection. The x-axis is a horizontal asymptote since $\lim_{x \to \infty} f(x) = 0$, and also $\lim_{x \to -\infty} f(x) = 0$.

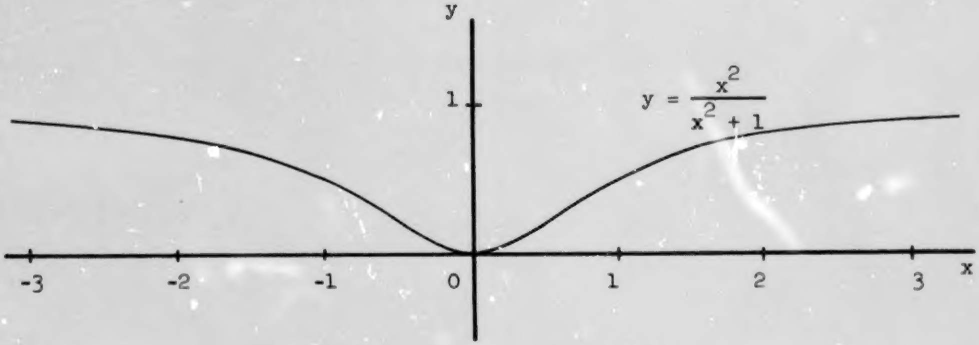


(b) $f(x) = \frac{x^2}{x^2 + 1}$

$f'(x) = \frac{2x}{(x^2 + 1)^2}$

$f''(x) = \frac{-2(3x^2 - 1)}{(x^2 + 1)^3}$

By Theorem 5-5a, $(0,0)$ is a minimum point. Points of inflection are $(\frac{\sqrt{3}}{3}, \frac{1}{4})$ and $(\frac{-\sqrt{3}}{3}, \frac{1}{4})$, by Definition 5-8. The line $y = 1$ is a horizontal asymptote, since $\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 1$.



5. Use information about extrema and flexure to draw the graph of

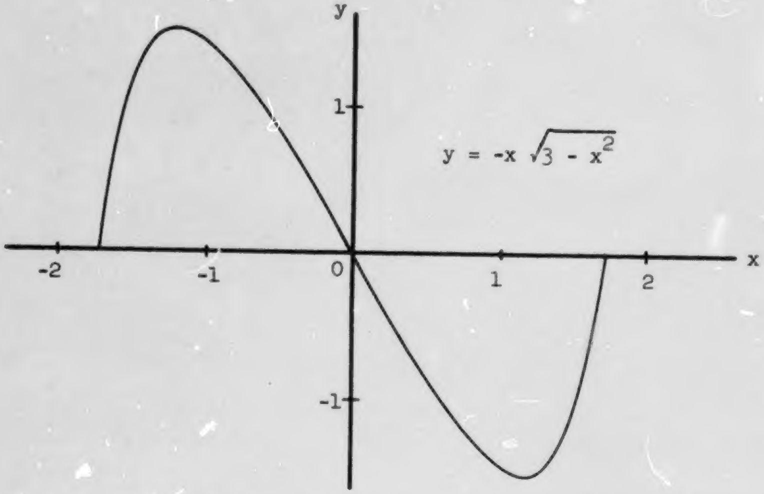
$$f(x) = -x\sqrt{3 - x^2}.$$

$f(x) = -x\sqrt{3 - x^2}$; domain: $|x| \leq \sqrt{3}$.

$$f'(x) = \frac{2x^2 - 3}{\sqrt{3 - x^2}}$$

$$f''(x) = \frac{x(9 - 2x^2)}{(3 - x^2)^{3/2}}.$$

Minimum point is at $\left(\sqrt{\frac{3}{2}}, -\frac{3}{2}\right)$ and maximum point is at $\left(-\sqrt{\frac{3}{2}}, \frac{3}{2}\right)$. The origin is a point of inflection. The graph of f is flexed downward for $-\sqrt{3} < x < 0$ and flexed upward for $0 < x < \sqrt{3}$.



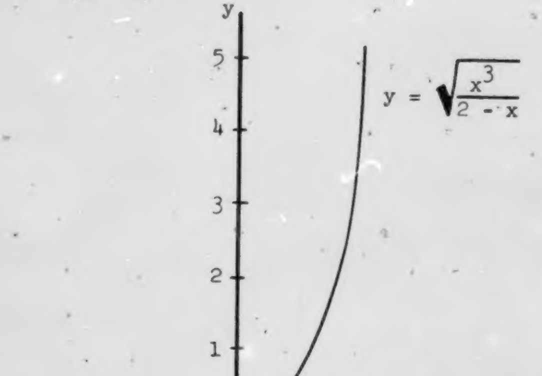
6. Draw the graphs of the given functions making use of extrema and information about flexure.

(a) $f : x \rightarrow \sqrt{\frac{x^3}{2 - x}}$.

Domain: $0 \leq x < 2$.

$$f'(x) = \frac{x^{1/2}(3 - x)}{(2 - x)^{3/2}}$$

$$f''(x) = \frac{6}{2x^{1/2}(2 - x)^{5/2}}$$

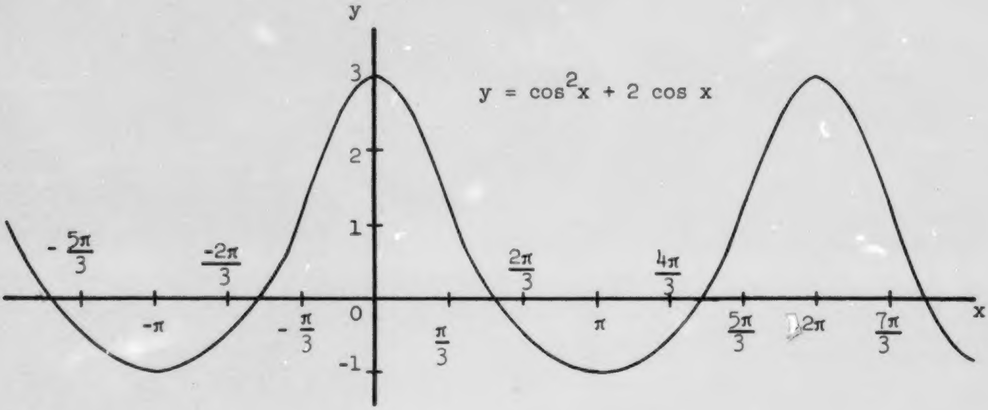


(b) $f : x \longrightarrow \cos^2 x + 2 \cos x$

$$f(x) = \cos^2 x + 2 \cos x$$

$$f'(x) = -2 \sin x(\cos x + 1)$$

$$f''(x) = -2(\cos x + 1)(2 \cos x - 1)$$



Maximum points at $x = 2k\pi$, k an integer.

Minimum points at $x = (2k + 1)\pi$, k an integer.

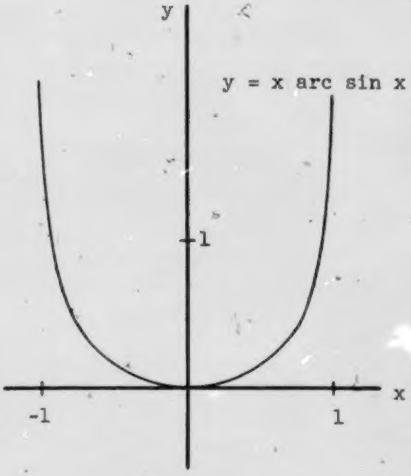
Points of inflection at $x = \pi(2k + \frac{1}{3})$ and at $x = \pi(2k + \frac{1}{3})$.

(c) $f : x \longrightarrow x \arcsin x$, $|x| < 1$.

$$f'(x) = \frac{x}{\sqrt{1 - x^2}} + \arcsin x$$

$$f''(x) = \frac{2 - x^2}{(1 - x^2)^{3/2}}$$

The graph is flexed upward for all x in the domain of definition. We have $f'(0) = 0$ and $f''(0) > 0$ so that $(0,0)$ is a minimum point.



7. Discuss symmetry, intercepts, asymptotes, extrema, intervals of flexure, and sketch the graph.

(a) $f(x) = \frac{2(x - 2)}{x^2}$.

Refer to Solutions Miscellaneous Exercises, Number 5.

$$f(x) = \frac{2(x - 2)}{x^2}$$

$$f'(x) = \frac{8 - 2x}{x^3}$$

$$f''(x) = \frac{4(x - 6)}{x^4}$$

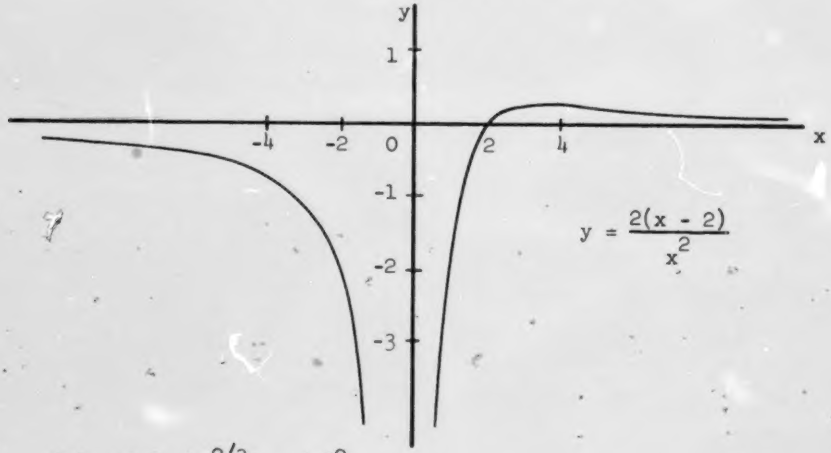
Intercepts: $f(x) = 0$ if and only if $x = 2$.

Horizontal asymptote: $y = 0$.

Vertical asymptote: $x = 0$.

Maximum point: $(4, \frac{1}{4})$.

$$\text{Flexure: } \begin{cases} \text{Curve is flexed downward for } x < 6 \text{ , } x \neq 0 \text{ , and} \\ \text{Curve is flexed upward for } x > 6 \text{ .} \end{cases}$$



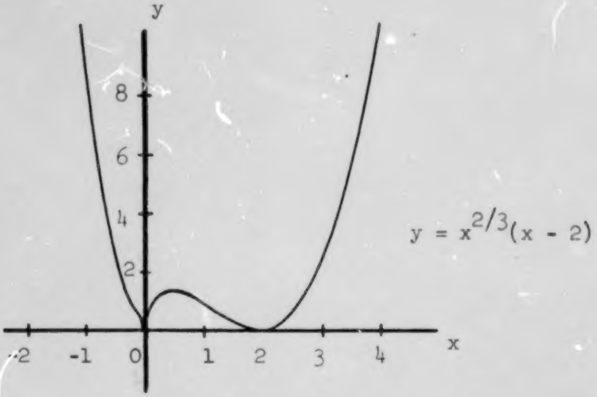
(b) $f(x) = x^{2/3}(x - 2)^2$

$$f'(x) = \frac{4(x - 2)(2x - 1)}{3x^{1/3}}$$

$$f''(x) = \frac{8(5x^2 - 5x - 1)}{9x^{4/3}}$$

The point $(\frac{1}{2}, \frac{9}{8}\sqrt[3]{2})$ is a local maximum; local minima are $(0,0)$ and

$(2,0)$. There is a cusp at $(0,0)$: $f'(x) \to -\infty$ as $x \to 0$ through positive values. There are points of inflection at $x = \frac{5 \pm 3\sqrt{5}}{10}$.



8. Draw the graph of $x^2y + xy^2 = 1$. Determine horizontal and vertical asymptotes, if any. Locate the axis of symmetry and the point of intersection of the curve with this line. Show that the curve has one and only one extremum and locate and classify this point.

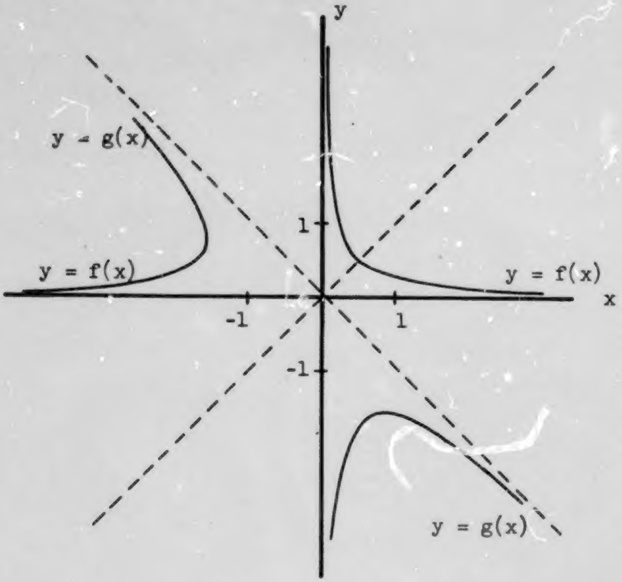
This relation has two branches given by

$$y = f(x) = \frac{-x + \sqrt{x^2 + \frac{4}{x}}}{2} \quad , \text{ domain } x > 0 \text{ and } x \leq \sqrt[3]{-4}$$

$$y = g(x) = \frac{-x - \sqrt{x^2 + \frac{4}{x}}}{2} \quad , \text{ domain } x > 0 \text{ and } x \leq \sqrt[3]{-4} .$$

As $x \to 0$ through positive values, f approaches $+\infty$ and g approaches $-\infty$. Since the relation is symmetric in x and y , the line $y = x$ is the axis of symmetry; thus, the line $y = 0$ is a horizontal asymptote. To find the extrema we set $y' = -\frac{y}{x}\frac{(y + 2x)}{(x + 2y)} = 0$, which implies $y + 2x = 0$ since the line $y = 0$ does not intersect the graph. Thus, upon substituting $y = -2x$ in the equation we obtain $(\frac{1}{2})^{1/3}$, $2(\frac{1}{2})^{1/3}$ as the only extremum. Computing y'' it can be shown that since $y'' < 0$ at this point, it is a maximum.

We observe that the graph of g has the line $y = -x$ as a slant asymptote; since $\lim\limits_{x \to \infty} g(x) + x = \lim\limits_{x \to -\infty} g(x) + x = 0$.



$$F(x,y) = x^2y + xy^2 - 1 = 0$$

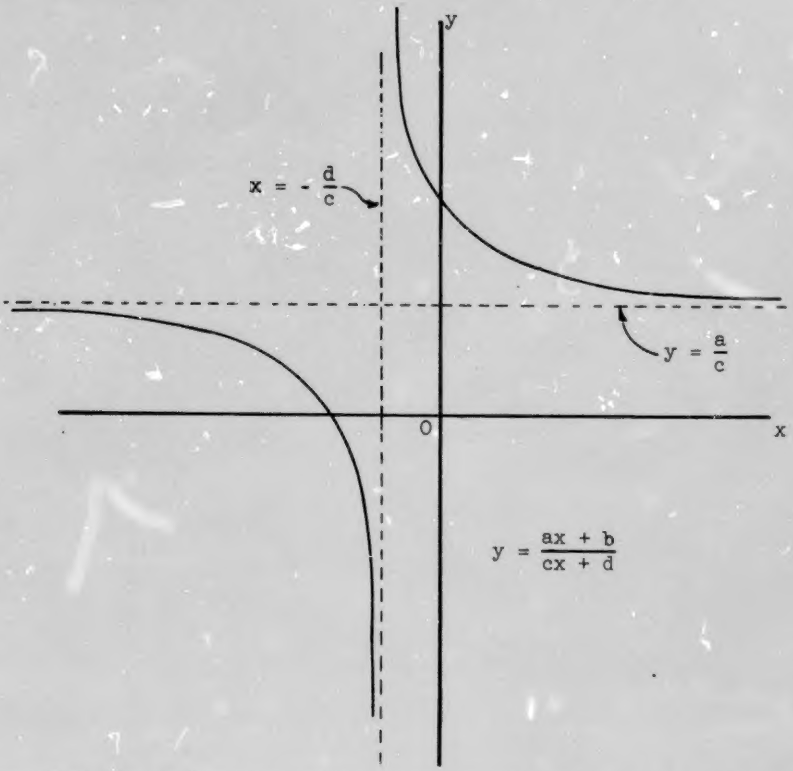
9. Show that the function $x \to \frac{ax+b}{cx+d}$ (assumed nonconstant) has no maxima or minima regardless of the values of a, b, c, and d.

If $c = 0$, the graph is a straight line and the result is valid. For $c \neq 0$,

$$\frac{ax+b}{cx+d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{c(x + \frac{d}{c})} .$$

Thus, the graph has a vertical asymptote at $x = -\frac{d}{c}$, and a horizontal asymptote at $y = \frac{a}{c}$. The graph is a hyperbola and is decreasing or increasing in $(\infty, -\frac{d}{c})$ and $(\frac{d}{c}, \infty)$ according to whether $b - \frac{ad}{c}$ is greater than or less than zero.

To verify that the graph is a hyperbola, we express it as $cxy + dy = ax + b$ or equivalently as $(x + \frac{d}{c})(y - \frac{a}{c}) = k$. We sketch the graph for the case where $b - \frac{ad}{c} > 0$.



10. Prove that the inflection points of the graph of $y = x \sin x$ lie on the curve $y^2(4 + x^2) = 4x^2$.

$y = x \sin x$

$y' = \sin x + x \cos x$

$y'' = 2 \cos x - x \sin x$ is zero if $2 \cos x = x \sin x$.

We know that $\sin x \neq 0$ since $\sin x = 0$ yields the contradiction $2 = 0$ in the equation $2 \cos x = x \sin x$. There is a point of inflection at

$x = \frac{2 \cos x}{\sin x}$, $y = x \sin x = 2 \cos x$. It follows that

$$\begin{aligned} y^2(4 + x^2) &= 4 \cos^2 x(4 + \frac{4 \cos^2 x}{\sin^2 x}) \\ &= 4 \frac{\cos^2 x}{\sin^2 x} (4 \sin^2 x + 4 \cos^2 x) \\ &= 16 \frac{\cos^2 x}{\sin^2 x} \\ &= 4(\frac{2 \cos x}{\sin x})^2 \\ &= 4x^2 \end{aligned}$$

A11. (a) Prove that if a curve is differentiable and flexed downward in an interval, the curve lies wholly under its tangent lines in this interval.

Let f be differentiable and flexed downward on $[a,b]$. For any u in $[a,b]$, the tangent line at $\big(u, f(u)\big)$ is given by

$$y = f'(u)(x - u) + f(u) .$$

We are to prove that $f(x) \leq y$, for all x in $[a,b]$. We have for any x in $[a,b]$,

$$\begin{aligned} f(x) - y &= f(x) - f(u) - f'(u)(x - u) \\ &= \Big(f'(z) - f'(u)\Big)(x - u) , \end{aligned}$$

for some z in $[a,b]$ between x and u, by the Law of the Mean. Since f is flexed downward on $[a,b]$, if $x < z < u$, then $f'(z) \geq f'(u)$; if $u < z < x$, then $f'(u) \geq f'(z)$, by Theorems 5-5b and 5-4a. It follows that

$$f(x) - y \leq 0 \qquad \text{for all } x \text{ in } [a,b] .$$

(b) Show that the graph of f crosses its tangent at $x = a$ if a is a point of inflection.

The tangent line at $x = a$ has equation

$$y = f'(a)(x - a) + f(a) .$$

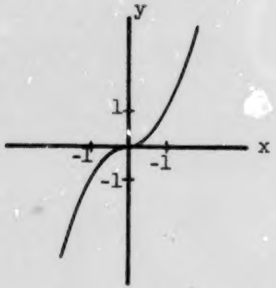
By Definition 5-8, f'' is strongly monotone in a neighborhood of a and $f''(a) = 0$. For $f''(x) > 0$, f' is increasing, by Theorem 5-4b. By the Law of the Mean,

$$\begin{aligned} f(x) - y &= f(x) - f(a) - f'(a)(x - a) \\ &= [f'(z) - f'(a)](x - a) , \end{aligned}$$

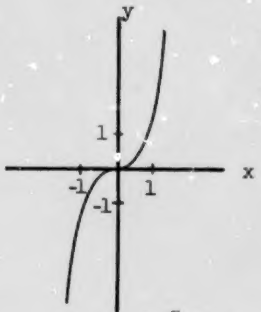
for some z between x and a; and $f(x) - y > 0$, by Theorem 5-4b, if $f''(x) > 0$. Similarly, if $f''(x) < 0$, $f(x) - y < 0$. It follows that the graph of f crosses the tangent line at $x = a$ since $f(a) = y$.

(c) Let $f(x) = x^n$ where n is a natural number. For what values of n, if any, does the graph have inflection points? Give sketches comparing the graphs for different values of n.

$f''(x) = n(n-1)x^{n-2}$ and thus the graph of f can have a point of inflection only at $x = 0$ (we assume $n > 2$). For n odd, $f''(x)$ changes sign at zero and therefore $(0,0)$ is a point of inflection. For n even, $f''(x) \geq 0$ for all x and $(0,0)$ is not an inflection point.



$y = x^3$



$y = x^5$

12. For a rational function given by

$$r(x) = \frac{a_p x^p + a_{p-1}x^{p-1} + \ldots + a_0}{b_q x^q + b_{q-1}x^{q-1} + \ldots + b_0}$$

where a_p, $b_q \neq 0$, find conditions for the existence of horizontal asymptotes, i.e., conditions that either limit, $\lim\limits_{x \to \infty} r(x)$ or $\lim\limits_{x \to -\infty} r(x)$, exists.

For $p < q$,

$$\lim_{x \to \infty} r(x) = \lim_{x \to \infty} \frac{a_p x^{p-q} + a_{p-1}x^{p-q-1} + \ldots + a_0 x^{-q}}{b_q + b_{q-1}x^{-1} + \ldots + b_0 x^{-q}}$$

$$= 0 .$$

For $p = q$,

$$\lim_{x \to \infty} r(x) = \lim_{x \to \infty} \frac{a_p + a_{p-1}x^{-1} + \ldots + a_0 x^{-q}}{b_q + b_{q-1}x^{-1} + \ldots + b_0 x^{-q}}$$

$$= \frac{a_p}{b_q} .$$

For $p > q$, $\lim r(x)$ does not exist since

$$r(x) = \frac{a_p + a_{p-1}x^{-1} + \dots + a_0^{-p}}{b_q x^{q-p} + \dots + b_0^{-p}}$$

increases without bound as $x \sim \infty$. The same conditions hold for the existence of $\lim\limits_{x \sim -\infty} r(x)$.

∧13. For what values of a does the function

$$f : x \longrightarrow \frac{x^2 + 2x + a}{x^2 + 4x + 3a}$$

assume all real values? Sketch the graph of the function for this case.

Since $f(x) = 1 - \frac{2(x + a)}{x^2+4x+3a}$, f will assume all real values if and only if g does, where

$$g(x) = \frac{x + a}{x^2 + 4x + 3a} .$$

Then

$$g'(x) = -\frac{x^2 + 2ax + a}{(x^2 + 4x + 3a)^2} .$$

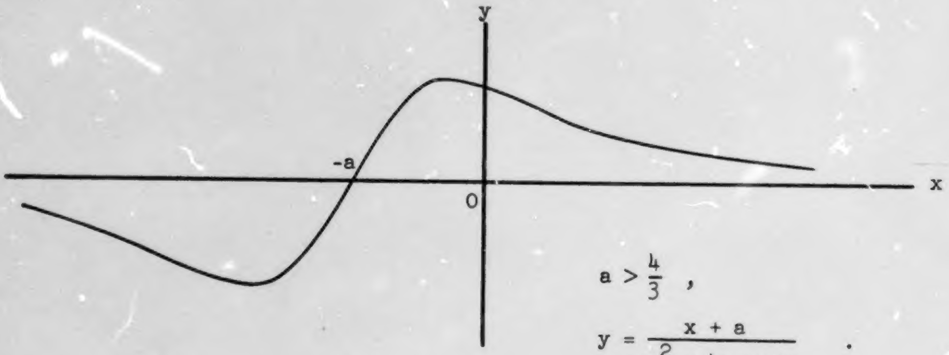
The graph of g depends qualitatively on whether $x^2 + 4x + 3a$ has zero, one, or two roots; but in all cases, $g(x)$ approximates 0 (and remains positive) as x gets large.

Case 1. $4 - 3a < 0$

Thus $x^2 + 4x + 3a$ has no real roots. Therefore g is continuous, and since $g(x)$ approximates zero for large positive and negative x, g is bounded above and below. Hence g cannot assume all values.

$$g'(x) = 0 \text{ when } x = -a \pm \sqrt{a(a-1)} .$$

This gives the extrema for g, hence for f.

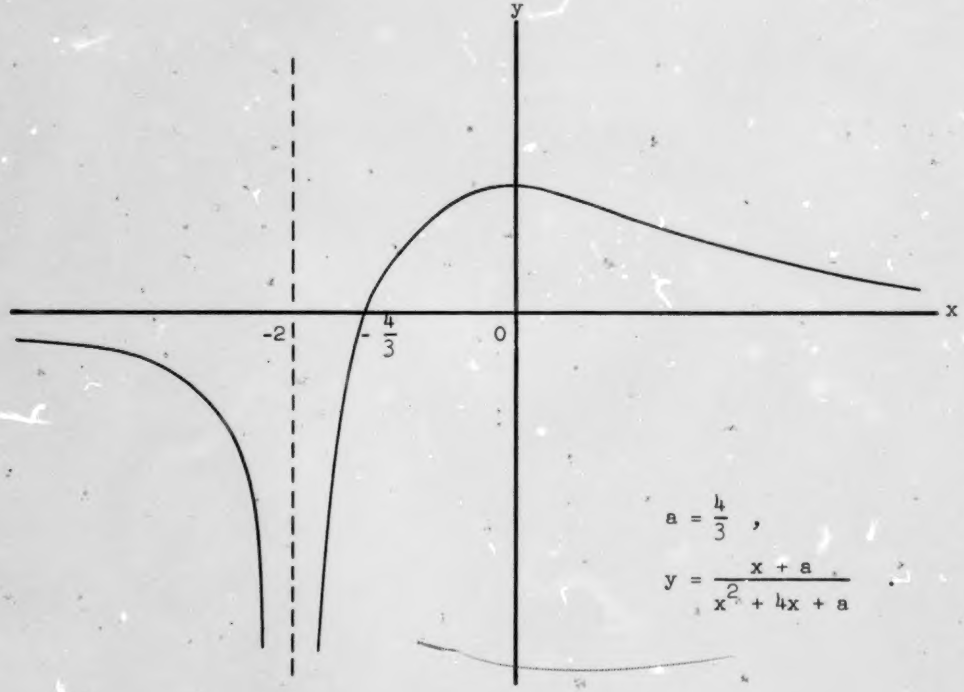


$a > \frac{4}{3}$,

$$y = \frac{x + a}{x^2 + 4x + 3a} .$$

Case 2. $4 - 3a = 0$

Thus $g(x) = \dfrac{x + \frac{4}{3}}{(x + 2)^2}$, and $g(-\frac{4}{3}) = 0$. $g(x) < 0$ for all $x < -\frac{4}{3}$, $g(x)$ approximates 0 for $|x|$ very large, and for $(x + 2)^2$ very small (i.e., for x close to -2), $-g(x)$ is very large. For $x > -\frac{4}{3}$, $g(x)$ has an upper bound by an argument similar to that in Case 1. So g does not take on all values.



$a = \frac{4}{3}$,

$$y = \frac{x + a}{x^2 + 4x + a} .$$

Case 3. $4 - 3a > 0$

First consider $a = 0$ and $a = 1$; here

$$g(x) = \frac{x}{x(x + 4)} , \qquad \text{for } a = 0 ,$$

and

$$g(x) = \frac{x + 1}{(x + 1)(x + 4)} , \qquad \text{for } a = 1 .$$

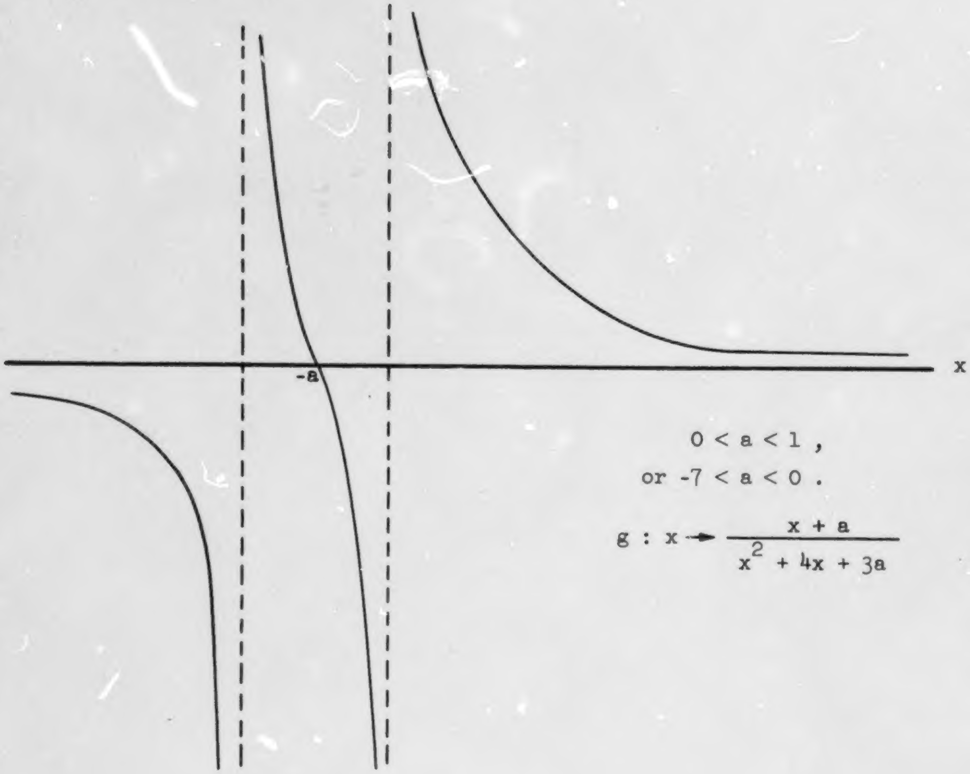
For both of these, $g(x) = 0$ for no x in the domain of g, and hence f cannot assume all values. For other values of a, $a < \frac{4}{3}$, we write $g(x)$ in the convenient form $g(x) = \dfrac{x + a}{(x+2+\sqrt{4-3a})(x+2-\sqrt{4-3a})}$

and note that the graph of g has vertical asymptotes at $x = -2 \pm \sqrt{4-3a}$. We divide Case 3 into three subcases: (i) the line $x = -a$ lies between these asymptotes; (ii) the line $x = -a$ lies to the right of both asymptotes; and (iii) the line $x = -a$ lies to the left of both asymptotes.

Subcase (i):

$$-2 - \sqrt{4-3a} < -a < -2 + \sqrt{4-3a}$$

if either $0 < a < 1$, or $-7 < a < 0$. As x approaches either root, the denominator of $g(x)$ approaches zero but remains either negative or positive, depending on whether x is approaching the root from inside or outside the region between the asymptotes. Therefore $g(x)$ changes sign for x close to and on different sides of an asymptote. For x between the asymptotes, $g(x) > 0$ for $x < -a$, $g(x) < 0$ for $x > -a$, and $g(x) = 0$ for $x = -a$. This characterizes the behavior of the graph of g near all asymptotes except the negative x-axis, where $g(x)$ approaches zero and remains negative. It is easy to see from the graph that g assumes all values.



$0 < a < 1$,

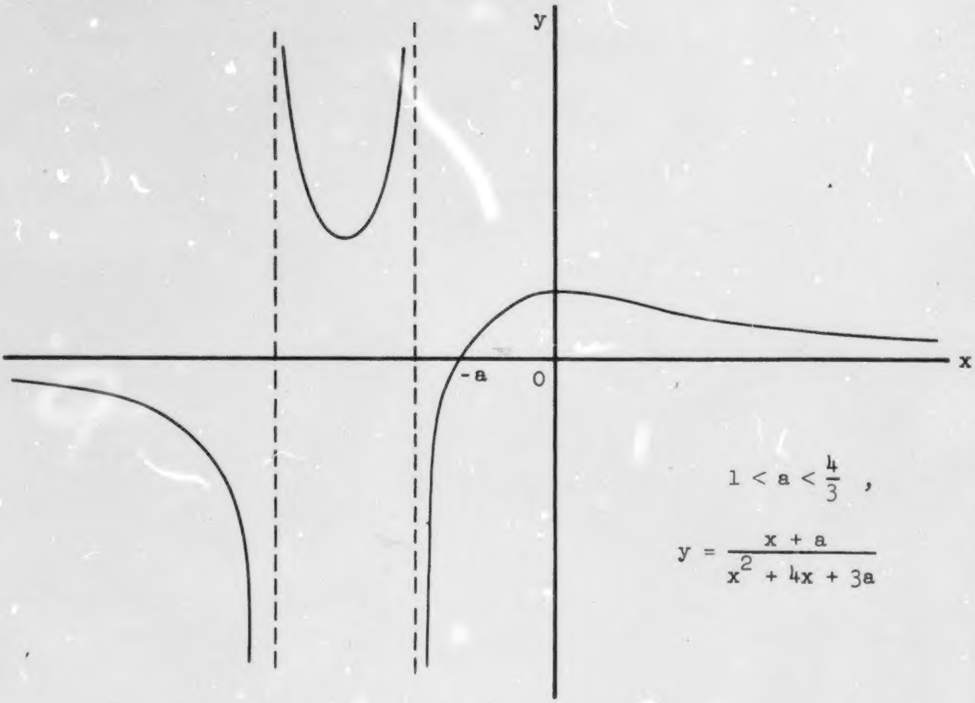
or $-7 < a < 0$.

$$g : x \to \frac{x + a}{x^2 + 4x + 3a}$$

Subcase (ii):

$$-2 + \sqrt{4-3a} < -a \quad \text{if} \quad 1 < a < \frac{4}{3} .$$

Here $g(x) > 0$ for all x between the asymptotes. As before, $g(x)$ changes sign for x close to and on different sides of either asymptote, and $g(x) \sim 0$ as $x \sim -\infty$, $g(x) < 0$. So g may or may not take on all values, depending on whether its local maximum is less than its local minimum.



$1 < a < \frac{4}{3}$,

$$y = \frac{x + a}{x^2 + 4x + 3a}$$

Subcase (iii):

By exactly the same arguments of Subcase (ii) it may be verified that a graph of Subcase (iii) may be obtained by turning the book upside-down and looking at the graph of Subcase (ii).

In each of our three cases, the graph of f can be obtained from that of g by a reflection about the x-axis followed by a scale change of 2 and an upward translation of 1.

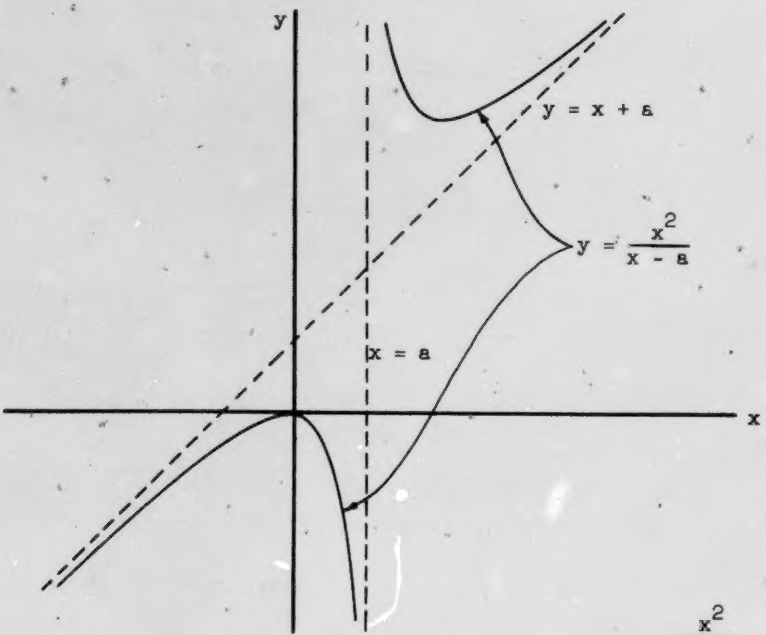
14. (a) Sketch the function

$$f : x \longrightarrow \frac{x^2}{x - a} , \quad a \neq 0 ,$$

and determine all the horizontal, vertical, and slant asymptotes.

$$\begin{aligned} f(x) &= \frac{x^2}{x - a} \\ &= x + a + \frac{a^2}{x - a} . \end{aligned}$$

As $|x|$ becomes very large, $f(x)$ becomes arbitrarily close to $x + a$. As $x \sim a^+$, $f(x)$ becomes arbitrarily large. As $x \sim a^-$, $-f(x)$ becomes arbitrarily large.



(b) In (a) the slope of the slant asymptote is $\lim\limits_{x \sim a} \frac{\frac{x^2}{x-a}}{x} = 1$. Show that for the function

$$f : x \longrightarrow 1 + x - 2\sqrt{x} \ , \quad x \geq 0 ,$$

the $\lim\limits_{x \sim \infty} \frac{f(x)}{x}$ exists although the graph has no linear asymptotes.

$$\frac{f(x)}{x} = \frac{1 + x - 2\sqrt{x}}{x} = 1 + \frac{1}{x} - \frac{2}{\sqrt{x}}$$

Therefore,
$$\lim_{x \sim \infty} \frac{f(x)}{x} = 1 .$$

So any linear asymptote would have to be of the form $x + b$. But for all b, $(x - b) - f(x)$ is arbitrarily large for x sufficiently large. For $x - b - (1 + x - 2\sqrt{x}) = -b - 1 + 2\sqrt{x}$, which is unbounded since $\sqrt{x}$ is unbounded.

Solutions Miscellaneous Exercises

1. Show that there are two tangent lines to the graph of $y = x^3 - 3x^2 + 3x$ which pass through the point $(4,1)$. Find their equations.

Let (x_0,y_0) be a point of tangency. Each line tangent to the curve y at the point (x_0,y_0) has slope $3(x_0 - 1)^2$. Each line through the points (x_0,y_0) and $(4,1)$ has slope $\frac{y_0 - 4}{x_0 - 1}$. Thus

$$3(x_0 - 1)^2 = \frac{y_0 - 1}{x_0 - 4} ,$$

whence $x_0 = 1$ or $x_0 = \frac{11}{2}$ (since $y_0 = x_0^3 - 3x_0^2 + 3x_0$). The line $y = 1$ is tangent at the point $(1,1)$ and the line $4y - 243x + 968 = 0$ is tangent at the point $(\frac{11}{2}, \frac{737}{8})$.

2. Show that the tangent line to the conic section

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

at a point (x_0,y_0) on the curve has the equation:

$$ax_0x + b(y_0x + x_0y) + cy_0y + d(x_0 + x) + e(y_0 + y) + f = 0 .$$

At the point (x_0,y_0) on the conic, the slope of the tangent line is

$$y' = -\frac{ax_0 + by_0 + d}{bx_0 + cy_0 + e} .$$

For the tangent line at the point (x_0,y_0),

$$\frac{y - y_0}{x - x_0} = -\frac{ax_0 + by_0 + d}{bx_0 + cy_0 + e} .$$

Thus the equation of the tangent line at (x_0,y_0) is

$$axx_0 + b(xy_0 + x_0y) + cyy_0 + d(x + x_0) + e(y + y_0) + f = 0$$

(using $ax_0^2 + 2bx_0y_0 + cy_0^2 + 2dx_0 + 2ey_0 = -f$).

A3. For what points (h,k) can one draw two tangent lines, one tangent line, no tangent line to the graph of

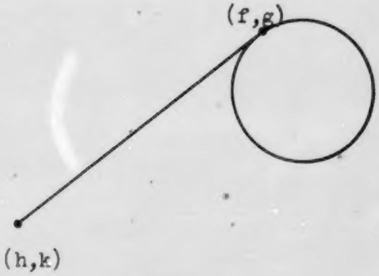
(a) $x^2 + 3xy + y^2 = a^2$, $(a > 0)$?

(b) $3x^2 + xy + 3y^2 = a^2$, $(a > 0)$?

(c) $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $(a > 0)$?

Interpret geometrically.

To find the equation of a tangent line from a point (h,k) to the curve $F(x,y) = 0$, we solve the simultaneous equations



$$(1) \quad \frac{y - h}{x - k} = \left.\frac{dy}{dx}\right|_{\substack{x=p \\ y=q}}$$

$$(2) \quad F(p,q) = 0 \ .$$

(Note that (1) merely states that the slope of the line joining points (h,k) and (x,y) is equal to the slope of the curve at (p,q).)

To find $\frac{dy}{dx}$ from $F(x,y) = 0$, it is generally easier to differentiate implicitly.

If we now eliminate either p or q , the resulting equation in the other variable will be a quadratic. The condition for two, one, or zero tangents will correspond to the condition for 2 real and unequal roots, 2 equal roots, or no real roots, respectively. This condition is satisfied by setting the discriminant greater than, equal to, or less than, zero, respectively.

Geometrically, the solution is easier once we recognize the curves. The curve

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

is a conic section.

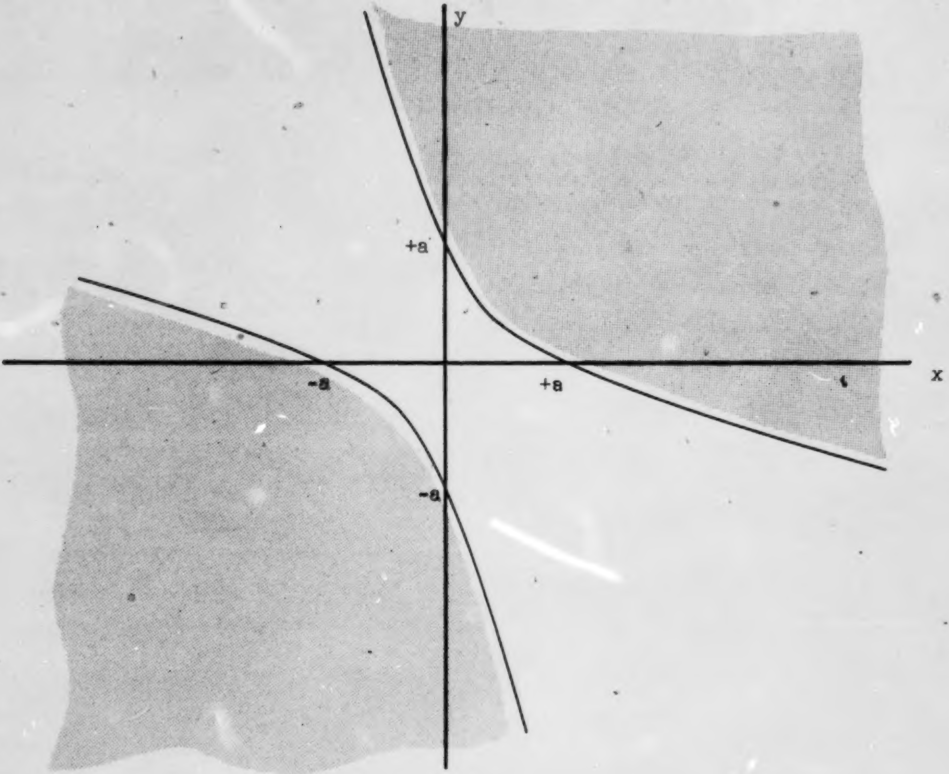
If $b^2 - 4ac > 0$, the curve is a hyperbola,

$b^2 - 4ac = 0$, the curve is a parabola,

$b^2 - 4ac < 0$, the curve is an ellipse.

(There can be degenerate cases of these e.g., the hyperbola case can include 2 intersecting straight lines, the elliptic case can include a single point $(x^2 + y^2 = 0)$).

(a) The discriminant, $b^2 - 4ac$, is 5; hence the curve is a hyperbola.

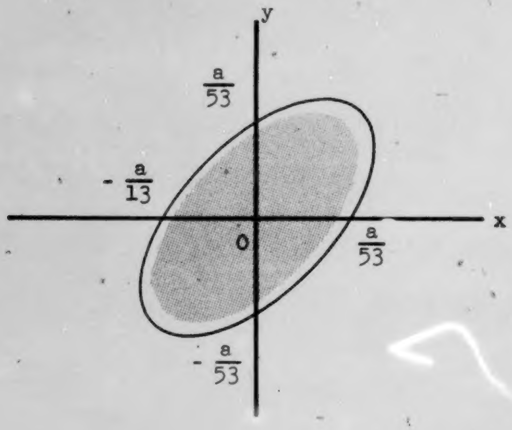


For any point (p,q) on the curve (that is, $a^2 - (p^2 + 3pq + q^2) = 0$) there will be exactly one tangent line to the curve.
For any point in the shaded region (i.e., $a^2 - (p^2 + 3pq + q^2) < 0$) there will be no tangent lines to the curve.

For any other points (that is, $a^2 - (p^2 + 3pq + q^2) > 0$) there will be two tangent lines to the curve.

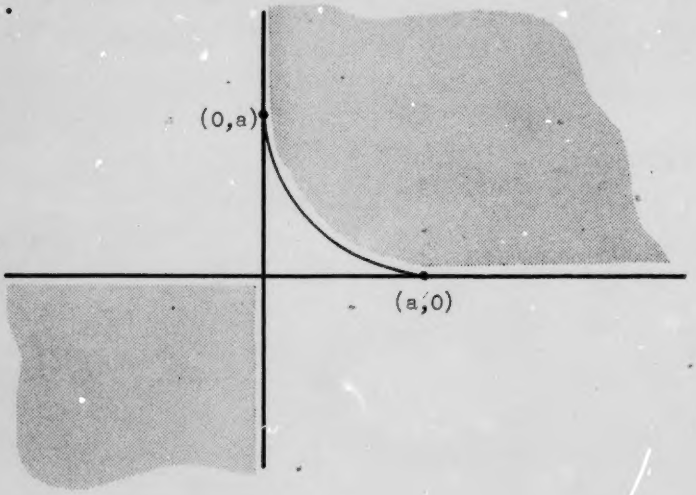
Note that the geometric interpretation agrees with results obtained algebraically by using the discriminant, $a^2 - (p^2 + 3pq + q^2) = 0$. The point $(0,0)$ is special in that the tangent lines to the curve which pass through $(0,0)$ are really asymptotes which are given by $y = mx$, where $m^2 + 3m + 1 = 0$.

(b) $b^2 = 4ac = -35$; curve is an ellipse.



For a point (p,q) on the curve (i.e., $3p^2 + pq + 3q^2 - a^2 = 0$) , There will be exactly one tangent line. For a point (p,q) inside the curve (i.e., $3p^2 + pq + 3q^2 - a^2 < 0$) , there will be no tangent lines to the curve. For a point outside the curve (where $3p^2 + pq + 3q^2 - a^2 > 0$) , there will be two tangent lines to the curve. Again $3p^2 + pq + 3q^2 - a^2$ corresponds to the discriminant.

(c) The curve here is a portion of a parabola whose axis is the line $y = x$.



For any point inside the cross-hatched region, there are no tangent lines to the curve.

For any point such that $\sqrt{x} + \sqrt{y} - \sqrt{a} \leq 0$, there are two tangent lines to the curve.

For any other point (i.e., all points in the second and fourth quadrants and the points $(x,0)$ and $(0,x)$, $x > a$ and $x < 0$), there is one tangent line to the curve.

If we had the entire parabola, i.e.,

$$\pm \sqrt{x} \pm \sqrt{y} = \sqrt{a} ,$$

then the conditions would be

$$\pm \sqrt{x} \pm \sqrt{y} - \sqrt{a} < 0 ,$$

$$\pm \sqrt{x} \pm \sqrt{y} - \sqrt{a} = 0 ,$$

$$\pm \sqrt{x} \pm \sqrt{y} - \sqrt{a} > 0 ,$$

for two, one, and zero tangents, respectively.

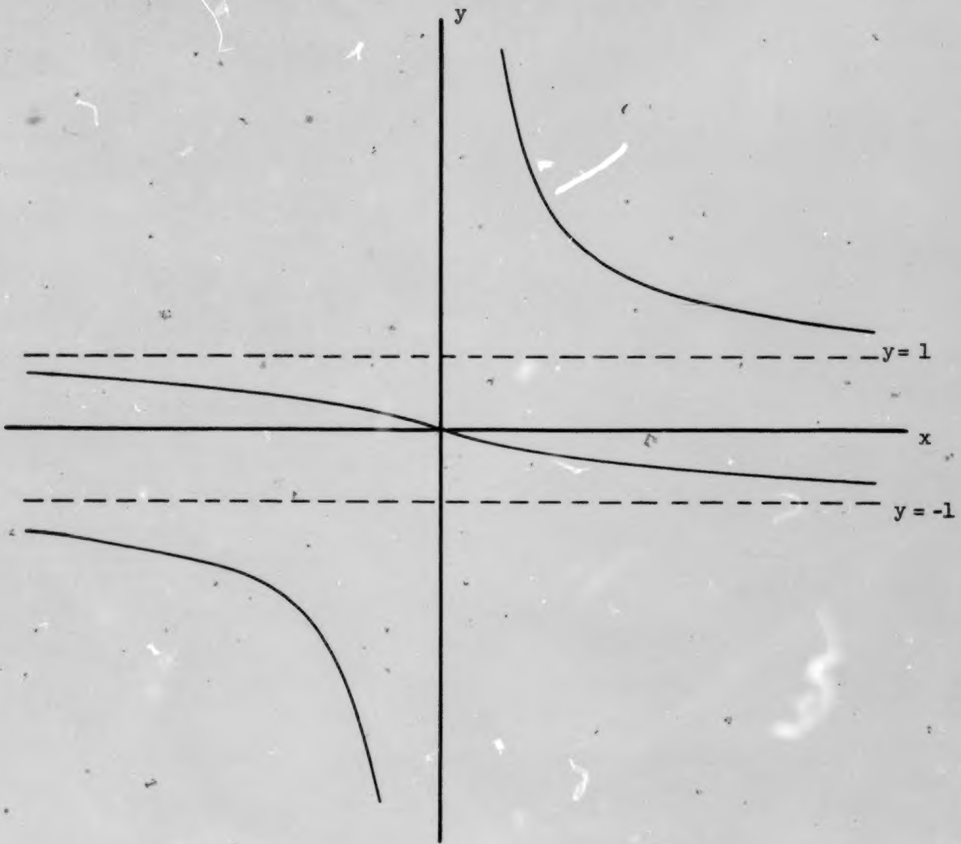
Also, in general, if we have a smooth closed convex curve, then there will be two , one, or zero tangent lines from a point, according to the point being outside, on, or inside the curve, respectively.

4. Determine equations of the horizontal and vertical asymtotes, if any, of the graph of
(a) $xy^2 - 4y - x = 0$.

Note: Refer to the discussion of Miscellaneous Exercises, Number 5.

The graph has the asymptote $x = 0$ for both large positive and negative y (since $\lim\limits_{y \to +\infty} \frac{4y}{y^2 - 1} = 0$ and $\lim\limits_{y \to -\infty} \frac{4y}{y^2 - 1} = 0$) .

It has the horizontal asymptote $y = 1$ to the right (since $\lim\limits_{y \to 1^+} \frac{4y}{y^2 - 1} = +\infty$) and to the left (since $\lim\limits_{y \to 1^-} \frac{4y}{y^2 - 1} = -\infty$) ; also the horizontal asymptote $y = -1$ to the right (since $\lim\limits_{y \to -1^+} \frac{4y}{y^2 - 1} = +\infty$) and to the left (since $\lim\limits_{y \to -1^-} \frac{4y}{y^2 - 1} = -\infty$) .
(See No. 5)



$$F(x,y) = xy^2 - 4y - x = 0$$

(b) $xy - \cos x = 0$

$$y = f(x) = \frac{\cos x}{x}$$

The graph of f has the vertical asymptote $x = 0$ for both large positive and negative y ; also, the horizontal asymptote $y = 0$ to the right and to the left.

5. Sketch the graphs of:

(a) $y = \dfrac{(x-a)^m}{(x-b)^n}$,

m , n integers, m , $n \geq 1$, $a \neq b$

(b) $y^2 = \dfrac{(x-a)^m}{(x-b)^n}$,

Λ(c) $y = \dfrac{(x-a)^m(x-b)^n}{(x-c)^p}$,

Λ(d) $y^2 = \dfrac{(x-a)^m(x-b)^n}{(x-c)^p}$,

m , n , p integers, m , n , $p \geq 1$, $a \neq b \neq c$.

In the discussion of this problem we do not attempt to consider each of the possible cases: m , n , p even or odd; $m > n > p$, $m < n < p$, etc.; $a > b > c$, etc. We are content to obtain a great deal of information about the graph of a rational function (and, thereafter, a relation) with minimum effort. In general, we examine the behavior of the graph for x in the neighborhood of a , b , c , respectively, and for large $|x|$. In particular, we consider the points where the curve "touches" (crosses, or is tangent to) the x-axis, horizontal asymptotes (to the right and/or to the left), and vertical asymptotes (above and/or below).

(a) $y = \dfrac{(x-a)^m}{(x-b)^n} = f(x)$

Case 1.

Consider the case: $a > b$, m even, n even, $m > n$.

Zeros of f : We set $x - a = 0$. The point $(a,0)$ is on the curve. Since m is even, $f(x)$ does not change sign in a neighborhood of a ; thus the graph of f is tangent to the x-axis at a . (If m were odd, the graph would cross the x-axis at a .)

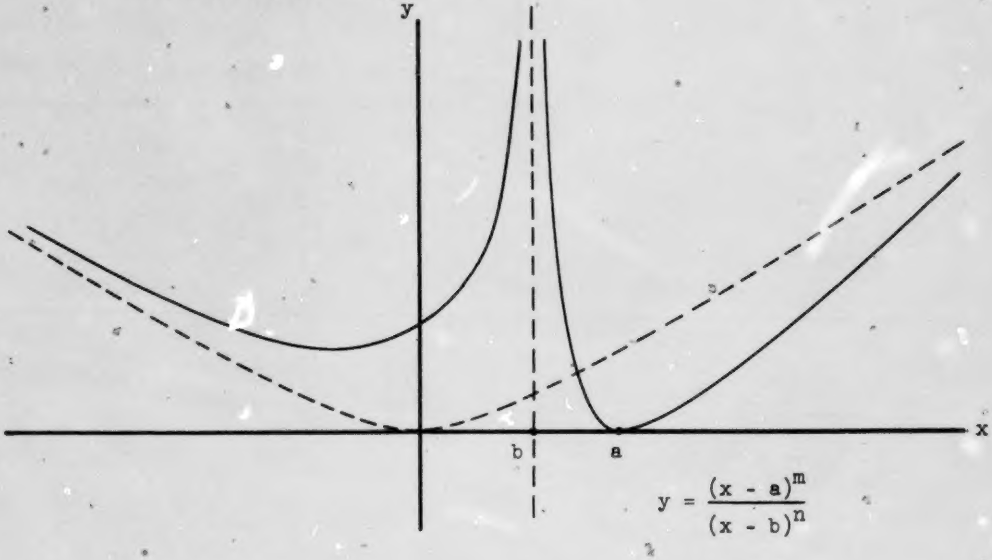
Vertical asymptote: Set $x - b = 0$. The line $x = b$ is a vertical asymptote since

$$\lim_{x \to b^-} \frac{(x-a)^m}{(x-b)^n} = +\infty \quad \text{and} \quad \lim_{x \to b^+} \frac{(x-a)^m}{(x-b)^n} = +\infty .$$

Behavior for large $|x|$: For large $|x|$, y approximates $\dfrac{x^m}{x^n} = x^{m-n}$. Thus the graph of f approximates the graph of $x \to x^{\alpha}$ $(\alpha = m - n > 0)$ for large $|x|$.

Sketch of graph: For the left branch of the curves $(x < b)$ we locate the minimum point by finding the zero of the derivative. The minimum is at $x = \frac{mb - na}{m - n}$, $m \neq n$.

Case 1. m , n even, $m > n$



Case 2.

Consider the case: $a < b$, m even, n odd, $m < n$.

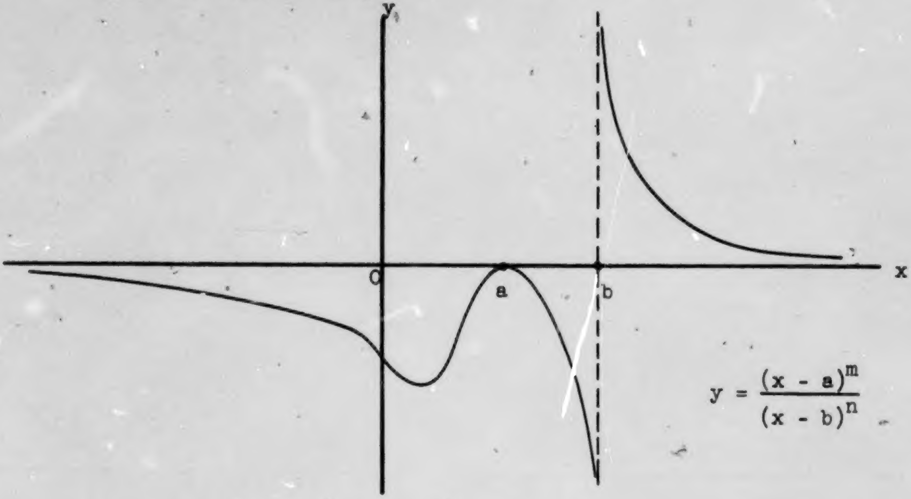
Zeros of f : See case 1.

Vertical asymptote: The line $x = b$ is a vertical asymptote above and below since

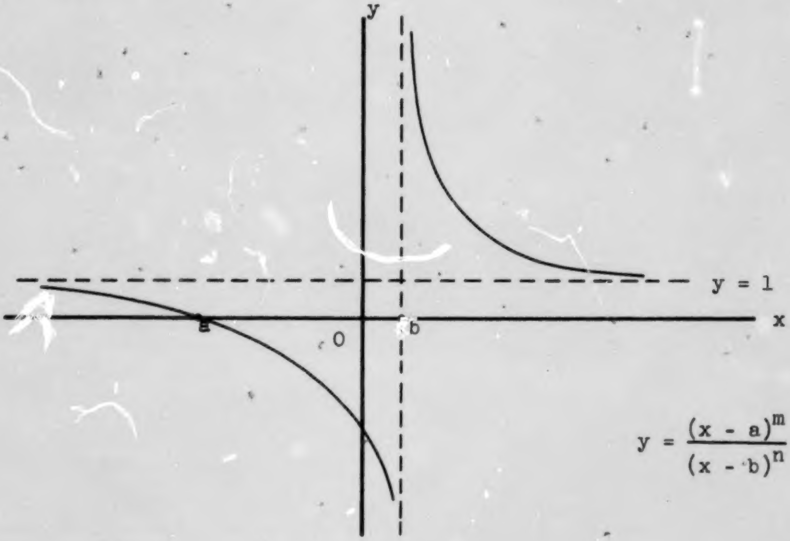
$$\lim_{x \to b^-} \frac{(x - a)^m}{(x - b)^n} = -\infty \quad \text{and} \quad \lim_{x \to b^+} \frac{(x - a)^m}{(x - b)^n} = +\infty .$$

Behavior for large $|x|$: For large $|x|$ y approximates $\frac{x^m}{x^n} = \frac{1}{x^{n-m}}$ $(n - m > 0)$. The x-axis is a horizontal asymptote to the right and to the left. Since the curve is tangent to the x-axis at $x = a$, it must have at least one minimum point for $x < a$. The minimum point is at $x = \frac{mb - na}{m - n}$.

Case 2. m even, n odd, m < n


y
x
0
a
b
$y = \frac{(x - a)^m}{(x - b)^n}$

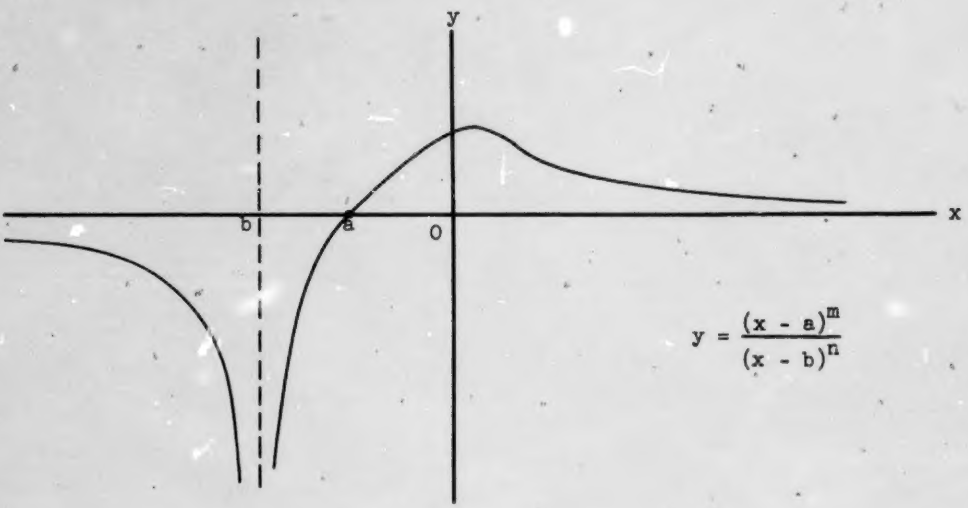
Case 3. m , n odd, m = n


y
y = 1
x
a
0
b
$y = \frac{(x - a)^m}{(x - b)^n}$

Case 4. $a > b$, m odd, n even, $m < n$.

In this case the maximum occurs at $x = \dfrac{mb - na}{m - n}$. (Compare with cases 1 and 2.)

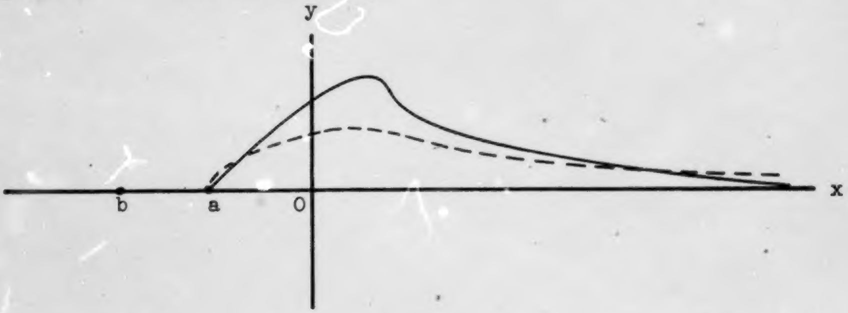
Case 4. m odd, n even, $m < n$



$$y = \frac{(x - a)^m}{(x - b)^n}$$

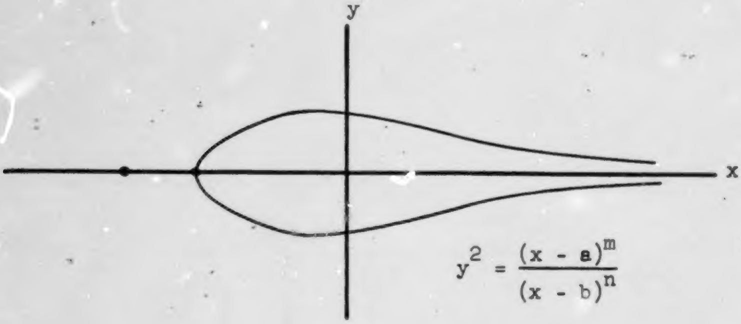
(b) $y^2 = \dfrac{(x - a)^m}{(x - b)^n}$

Consider the case: $a > b$, m odd, n even, $m < n$.
The portion of the graph for $x \geq a$ in part (a) , case 4, is the graph of y^2 (the solid curve).



Then, from the graph of y^2 we obtain (by approximating the square root of ordinates) the required graph of $|y|$ (the dotted curve) and, thereafter, by reflection in the x-axis, the required graph.

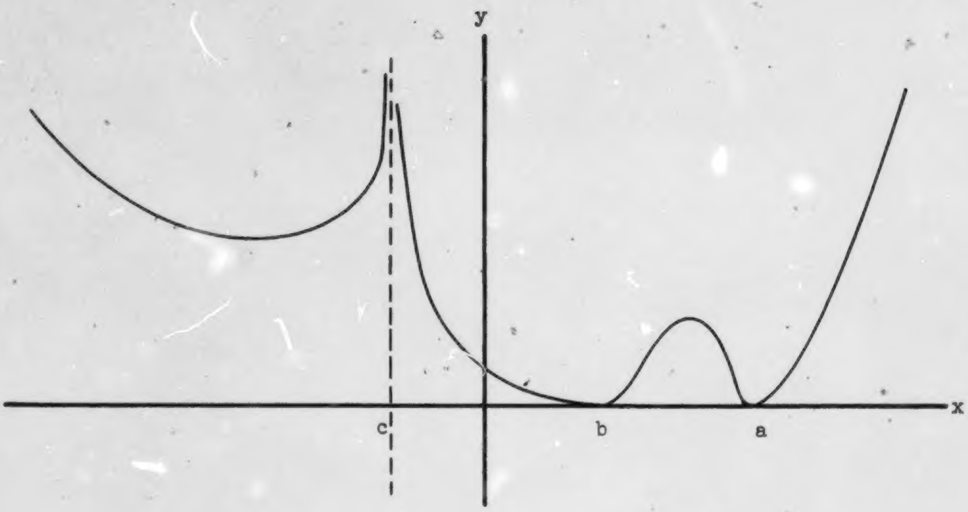
The curve is symmetric with respect to the x-axis.



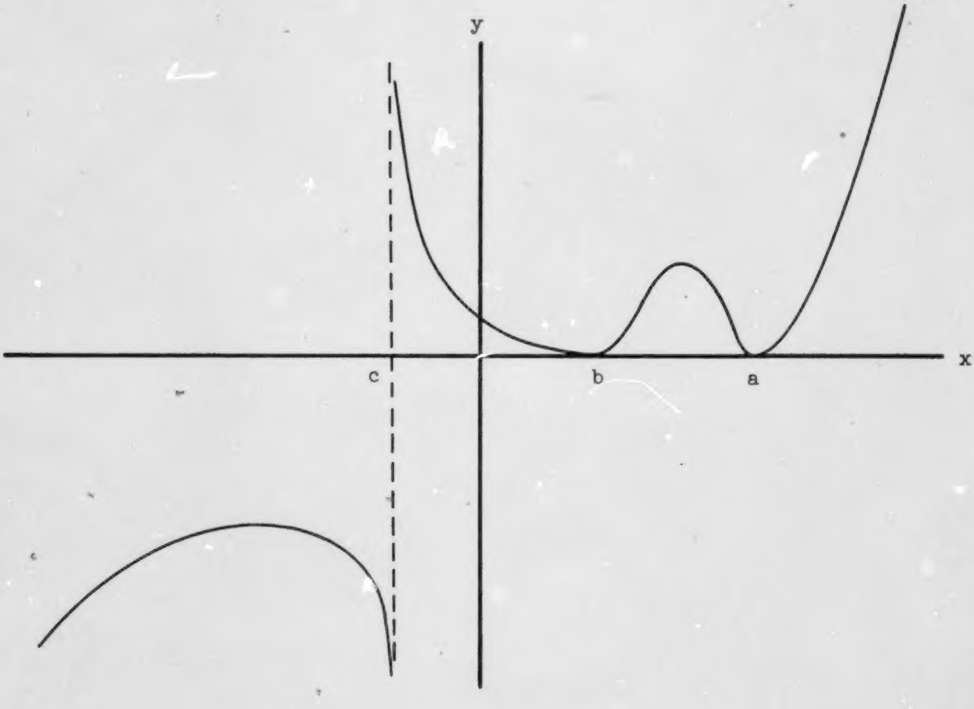
A (c) $y = \frac{(x-a)^m (x-b)^n}{(x-c)^p}$

Let $a > b > c$.

We observe that for m, n, p even, $m + n \geq p$, the situation is like that in (a), case 1. The line $x = c$ is an asymptote above for each branch of the curve (since m, n, p are even). Again, the curve is tangent to the x-axis at $x = a$, and in addition, at $x = b$. For large $|x|$ the curve approximates the graph of $x \rightarrow x^{\alpha}$, where $\alpha = m + n - p > 0$.



For m , n even, p odd, $m + n > p$, the left branch of the curve is reflected in the x-axis.



In general, if $m + n = p$, the curve approximates the line $y = 1$; i.e., the line $y = 1$ is a horizontal asymptote. If $m + n < p$, the x-axis is a horizontal asymptote.

(d) $y^2 = \dfrac{(x - a)^m(x - b)^n}{(x - c)^p}$

As in part (b) we may use the results (in respective cases) of (c) to obtain the graph of y^2 and thereafter the graph of $|y|$. Finally, by reflecting the graph of $|y|$ in the x-axis, we obtain the required graph.

6. In past SMSG writers' conferences it has been observed that when the number of writers on a team is 28 or greater, all the available time is spent in discussion between members of the group, so that no writing gets done. Assuming that in a group of x writers $(28 \geq x \geq 1)$ each participant is engaged in writing $40\left(1 - (\frac{x - 1}{27})^2\right)$ hours per week, determine the size of the team which maximizes the total number of hours the group spent writing. (Draw your own conclusions about the team which wrote this book).

The total number of hours worked by the team is given by

$f(x) = 40kx\left(1 - (\frac{x - 1}{27})^2\right)$, where k is the number of weeks worked, k independent of x. Then $f'(x) = 40k\left(1 - (\frac{x - 1}{27})^2 - \frac{2x(x - 1)}{27^2}\right)$.

$f'(x) = 0$ if $3x^2 - 4x - 728 = 0$; i.e., if $x = \dfrac{4 + \sqrt{16 + 8736}}{6}$.

Since we are interested in an integral solution, and since $16 < \dfrac{4 + \sqrt{8752}}{6} < 17$, we consider $x = 16$ and $x = 17$. By Corollary 2 to Theorem 5-2b, since $f(0) = f(28) = 0$ and $f(16) > f(17) > 0$, a team of 16 members is the most efficient team.

7. A picture h feet high is placed on a wall with its base b feet above the level of the observer's eye. If he stands x feet from the wall, verify that the angle of vision ϕ subtended by the picture is given by

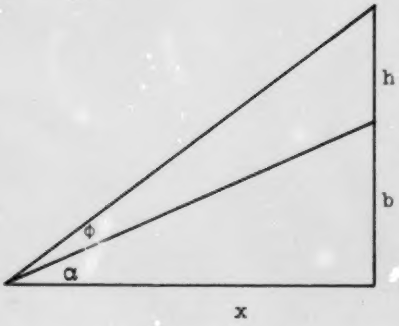
$$\phi = \operatorname{arccot} \frac{x}{h + b} - \operatorname{arccot} \frac{x}{b}.$$

Show that to get the "best" view of the picture, i.e., the largest angle of vision, the boserver should stand $\sqrt{b(h + b)}$ feet away from the wall.

Since $\phi + \alpha = \operatorname{arccot} \frac{x}{h + b}$

and $\alpha = \operatorname{arccot} \frac{x}{b}$,

$\phi = \operatorname{arccot} \frac{x}{h + b} - \operatorname{arccot} \frac{x}{b}$.



We have

$$D_x\phi = \frac{-(h + b)}{(h + b)^2 + x^2} + \frac{b}{b^2 + x^2}$$

$$= \frac{bh(h + b) - hx^2}{[(h + b)^2 + x^2](b^2 + x^2)}$$

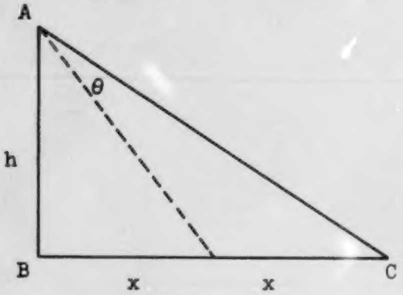
whence $D_x\phi$ is zero if

$$bh(h + b) = hx^2$$

or $$x = \sqrt{b(h + b)} .$$

By Theorem 5-4c, ϕ is maximum at $x = \sqrt{b(x + b)}$.

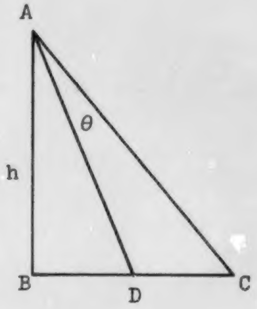
8. Let ABC be a right triangle with AB perpendicular to BC , the length of AB = h , the length of BC = 2x . Let AD be the median to side BC . Determine x so that the angle θ between the median and the hypotenuse of triangle ABC is a maximum.



$$AD = \sqrt{x^2 + h^2}$$

and $$AC = \sqrt{h^2 + 4x^2} ,$$

whence by applying the Law of Cosines to triangle ABC we have,



$$x^2 = (x^2 + h^2) + (h^2 + 4x^2) - 2\sqrt{x^2 + h^2} \cdot \sqrt{4x^2 + h^2} \cdot \cos \phi ,$$

or $$\cos \phi = \frac{h^2 + 2x^2}{[(x^2 + h^2)(4x^2 + h^2)]^{1/2}} .$$

It follows that

$$\frac{d\phi}{dx} = \frac{-4x(4x^4 + 5x^2h^2 + h^4) + (h^2 + 2x^2)(5xh^2 + 8x^3)}{\sin\phi\,(4x^4 + 5x^2h^2 + h^4)^{3/2}}$$

is zero if $16x^4 + 20x^2h^2 + 4h^4 = 5h^4 + 18x^2h^2 + 16x^4$, or $x = \dfrac{h}{\sqrt{2}}$.

By Theorem 5-4c the angle ϕ is a maximum if $x = \dfrac{h}{\sqrt{2}}$.

9. The location of an object on a straight line at time t , is given by the formula

$$S = At - (1 + A^4)t^2 .$$

Show that the object moves forward initially, when A is positive, but ultimately retreats. Show also that for different values of A the maximum possible distance that the particle can move forward is $\frac{1}{8}$.

$\dfrac{dS}{dt} = A - 2(1 + A^4)t$, so that when $t = 0$, $\dfrac{dS}{dt} = A$ which is positive.

The object moves forward, i.e., $\dfrac{dS}{dt}$ is positive, for $\dfrac{A}{2(1 + A^4)} > t$.

The object retreats ($\dfrac{dS}{dt}$ is negative) for $t > \dfrac{A}{2(1 + A^4)}$. For a fixed value of A the particle reaches its maximum forward distance $S = \dfrac{A^2}{4(1 + A^4)}$ at $t = \dfrac{A}{2(1 + A^4)}$. For each value of A the maximum forward distance of $S = \dfrac{A^2}{4(1 + A^4)} = f(A)$ is $\frac{1}{8}$. This follows since $f'(A) = \dfrac{A(1 - A^4)}{2(1 + A^4)^2}$ is zero if $A = 1$, and (by Theorem 5-4c) $f(1) = \frac{1}{8}$ is a maximum value of f .

10. A man standing at the edge of a circular swimming pool wishes to reach a point $\frac{1}{4}$ of the way around the pool in the least possible time. He plans to run along the edge of the pool for some distance and then swim straight to his destination. If he can swim 20 feet per second and run 24 feet per second, how far should he run before diving in?

Let v represent the distance the man swims and u , the distance he runs: the total distance is u + v and the time t required is given by

$$t = \frac{u}{24} + \frac{v}{20} .$$

Let r represent the radius of the pool. Then

$$u = \frac{\pi}{2}r - r\theta ,$$

$$= r(\frac{\pi}{2} - \theta) ,$$

and

$$v^2 = (r \sin \theta)^2 + (r - r \cos \theta)^2$$

$$= 2r^2 - 2r^2 \cos \theta$$

$$= 4r^2 \sin^2 \frac{\theta}{2} .$$

Differentiating, we obtain

$$\frac{dt}{d\theta} = \frac{1}{24}\frac{du}{d\theta} + \frac{1}{20}\frac{dv}{d\theta} ,$$

$$\frac{dv}{d\theta} = r \cos \frac{\theta}{2} ,$$

$$\frac{du}{d\theta} = -r ,$$

whence

$$\frac{dt}{d\theta} = \frac{-r}{24} + \frac{r}{20} \cos \frac{\theta}{2}$$

and

$$\frac{d^2t}{d\theta^2} = - \frac{r}{40} \sin \frac{\theta}{2} .$$

Since $\frac{d^2t}{d\theta^2}$ is less than zero for $0 < \theta < \frac{\pi}{2}$, the curve is flexed downward (Theorem 5-5b) so that minimum points must be endpoints.

When $\theta = 0 , t = \frac{\pi r}{48}$,

and when $\theta = \frac{\pi}{2} , t = \frac{\sqrt{2}}{20}r$.

Since $\frac{\sqrt{2}}{20}r > \frac{\pi r}{48}$, the time is minimum when $\theta = 0$. The man should run all the way and not swim at all.

A11. A conical cup with radius r, height h, is filled with water. Find the radius R of the sphere which displaces the largest volume of water when jammed into the cup.

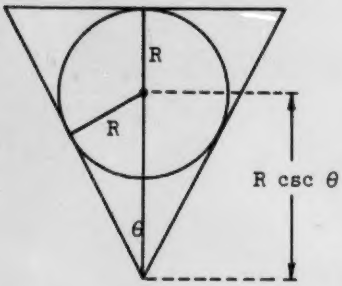
There are three cases to consider:

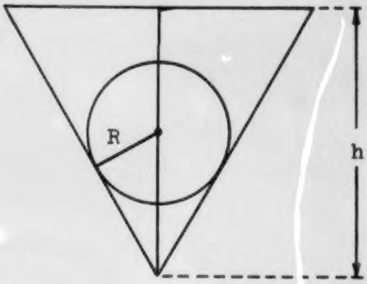
Case 1. The sphere is completely submerged in the cone. For this case, the largest displacement occurs (see figure below) when

$$R + R \csc \theta = h$$

or

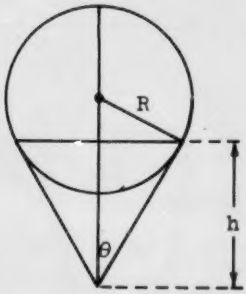
$$R = \frac{h}{1 + \csc \theta} .$$





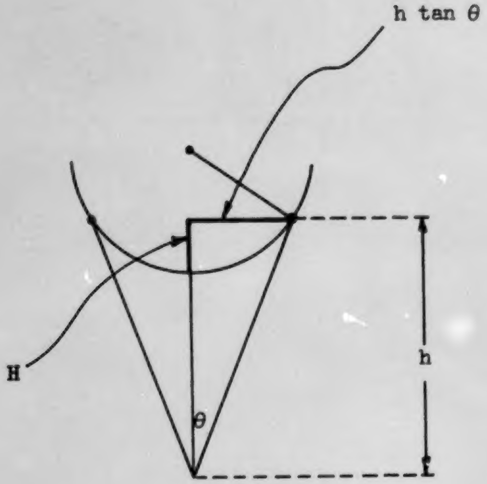
Case 2. The sphere is tangent to the cone and partially submerged. In this case,

$$R = h \sec \theta \tan \theta .$$



Case 3. The sphere is sitting on the cone. In this case,

$$R > h \sec\theta \tan\theta \ .$$



Set H as the height of the spherical segment (see figure) submerged in water. Then

$$H = R - \sqrt{R^2 - h^2 \tan^2\theta}$$

whence
$$R = \frac{H^2 + h^2 \tan^2\theta}{2H} \ .$$

Using the formula for the volume V of the segment of a sphere of radius R and height H ,

$$V = \frac{\pi}{3}H^2(3R - H) \ ,$$

we have

$$V = \frac{\pi}{6}H(H^2 + 3h^2 \tan^2\theta) \ .$$

Thus, for maximum volume, V , H should be as large as possible. From above, since

$$H = R - \sqrt{R^2 - h^2 \tan^2 \theta}$$

$$= \frac{h^2 \tan^2 \theta}{R + \sqrt{R^2 - h^2 \tan^2 \theta}} \ ,$$

to maximize H we make R as small as possible $(R \geq h \tan \theta)$. Consequently, the volume V is greater in Case 2 than in Case 3.

We now compare the volume for Cases 1 and 2.

Let V_1 and V_2 represent the volume in Cases 1 and 2, respectively. Then,

$$V_1 = \frac{4}{3}\pi R^3$$

$$= \frac{4}{3}\pi \left(\frac{h}{1 + \csc \theta}\right)^3 .$$

$$V_2 = \frac{\pi}{6} H (H^2 + 3h^2 \tan^2 \theta) .$$

where

$$H = \frac{h^2 \tan^2 \theta}{(h \sec \theta \tan \theta) + \sqrt{(h^2 \sec^2 \theta \tan^2 \theta) - h^2 \tan^2 \theta}}$$

$$= \frac{h \tan \theta}{\sec \theta + \tan \theta}$$

(where $\tan \theta > 0$ since $0 < \theta < \frac{\pi}{2}$).

Thus

$$V_2 = \frac{\pi}{6} h^3 \cdot \frac{\tan^3 \theta}{\sec \theta + \tan \theta} \cdot \left(\frac{1}{(\sec \theta + \tan \theta)^2} + 3\right) .$$

Now we have

$$\left(\frac{1}{1 + \csc \theta}\right)^3 = \frac{\tan^3 \theta}{(\tan \theta + \sec \theta)^3} .$$

Also,

$$\sec \theta + \tan \theta = \frac{1 + \sin \theta}{\cos \theta} > 1$$

in the domain of θ, so that

$$\frac{1}{(\sec \theta + \tan \theta)} < 1 .$$

$$V_2 < \frac{\pi h^3}{6} \frac{\tan^3 \theta}{\sec \theta \tan \theta} \cdot \frac{4}{(\sec \theta + \tan \theta)^2}$$

$$\leq \frac{4\pi h^3}{6} \left(\frac{\tan \theta}{\sec \theta + \tan \theta}\right)^3$$

$$\leq \frac{4\pi h^3}{6} \left(\frac{1}{1 + \csc \theta}\right)^3$$

$$\leq \frac{V_1}{2} \, .$$

So for maximum volume, V ,

$$R = \frac{h}{1 + \csc \theta}$$

$$= \frac{h}{1 + \frac{\sqrt{r^2 + h^2}}{r}}$$

$$= \frac{rh}{r + \sqrt{r^2 + h^2}} \, .$$

Teacher's Commentary

Appendix 1

THE REAL NUMBERS

We treat the real numbers axiomatically in order to provide a skeleton summary of their properties. The goal is review not systematic development of the already familiar properties of the real numbers from a set of axioms.

Solutions Exercises A1-1

1. Verify:
   (a) that the natural numbers are not closed under subtraction and division:

   Consider, for example, $2 - 5$ and $2 \div 5$.

   (b) that the rational numbers form a field.

   Treat the rationals as a subset of the real number field. Then it is sufficient to demonstrate that the rationals are closed under addition and multiplication and that they have inverses under these operations.

   (c) that the operations of subtraction and division on the reals are not commutative or associative.

   Consider, for example, the following:

$$5 - 7 = -2 \neq 2 = 7 - 5$$
$$4 \div 2 = 2 \neq \frac{1}{2} = 2 \div 4$$
$$7 - (5 - 3) = 7 - 2 = 5 \neq -1 = 2 - 3 = (7 - 5) - 3$$
$$8 \div (4 \div 2) = 8 \div 2 = 4 \neq 1 = 2 \div 2 = (8 \div 4) \div 2$$

2. **Prove:** For any real number a , $-(-a) = a$.

Every element has an additive inverse, and the sum of these is equal to zero. Take the element $-a$.
Then,

$$(-a) + \big(-(-a)\big) = 0 ,$$

and

$$a + (-a) + \big(-(-a)\big) = a + 0 .$$

By field properties:

$$\big(a + (-a)\big) + \big(-(-a)\big) = 0 + \big(-(-a)\big) = -(-a) = a .$$

3. **Prove:** For any real number a , $(-1)a = -a$.

$$(-1)a + (1)a = (-1 + 1)a = 0 \cdot a$$

or

$$(-1)a + a = 0 .$$

Hence $(-1)a$ is the additive inverse of a ; i.e., $(-1)a = -a$.

4. **Prove:** For any real numbers a and b , $-(a + b) = (-a) + (-b)$.

From No. 3,

$$-(a + b) = (-1)(a + b) = (-a) + (-b) .$$

5. **Prove:** For any real numbers a and b , $(-a)(-b) = ab$.

$$\begin{aligned} 0 &= (-a)\big(b + (-b)\big) = (-a)(b) + (-a)(-b) \\ &= (-1)ab + (-a)(-b) \\ &= -ab + (-a)(-b) \end{aligned}$$

Hence, $(-a)(-b)$ is the additive inverse of $-ab$; i.e., $(-a)(-b) = ab$.

6. **Prove:** For any real numbers a and b , $ab = 0$, if and only if $a = 0$ or $b = 0$ (or $a = b = 0$) .

Suppose $b \neq 0$ and $ab = 0$, then,

$$a = a \cdot 1 = a \cdot (b \cdot \tfrac{1}{b}) = ab(\tfrac{1}{b}) = 0 \cdot (\tfrac{1}{b}) = 0 .$$

So $a = 0$. In the same way, it can be shown that $b = 0$, if $a \neq 0$. If $a = 0$, $ab = 0 \cdot b = 0$ and there is no restriction on b , so b may also equal zero.

7. Verify that the numbers $\frac{a + b}{2}$, where a and b are rational numbers, constitute a field.

This follows immediately from Number 1 (b).

Solutions Exercises A1-2

1. Prove: For any real number a :
(a) if $a > 0$, then $0 > -a$.
(b) if $0 > a$, then $-a > 0$.

(a)
$$a > 0$$
$$a + (-a) > 0 + (-a)$$
$$0 > -a$$

(b)
$$0 > a$$
$$0 + (-a) > a + (-a)$$
$$-a > 0$$

2. Prove: For any real numbers a , b , c , d if $a > b$ and $c > d$, then $a + c > b + d$.

Since $a > b$ and $c > d$,

$a + c > b + c$ and $b + c > b + d$.

Therefore, by transitive law: $a + c > b + d$.

3. Prove: For any real numbers a , b , c if $a > b$ and $c < 0$, then $bc > ac$.

$$c < 0 \Longrightarrow -c > 0 ;$$
$$a > b \text{ and } -c > 0 \Longrightarrow a(-c) > b(-c) .$$

Adding $bc + ac$ to both members of the inequality gives $bc > ac$.

4. (a) For any positive numbers a and b , $a > b$ if and only if $a^2 > b^2$.

Suppose $a > b$. Then $a^2 > ab$ and $ab > b^2$. Hence, $a^2 > b^2$. Conversely, suppose $a^2 > b^2$. By trichotomy, $a > b$, $a = b$, or $b > a$. But $a = b \Longrightarrow a^2 = b^2$, so $a \neq b$. Also $b > a \Longrightarrow b^2 > a^2$ since a and b are positive. So, $b \not> a$. Hence, $a > b$.

(b) For any negative numbers a and b, $a > b$ if and only if $b^2 > a^2$.

Suppose $a > b$. Then $-b > -a$ and $-b$ and $-a$ are positive. Hence, from 4 (a), $(-b)^2 > (-a)^2$ or $b^2 > a^2$. Conversely, suppose $b^2 > a^2$. Since $b^2 = (-b)^2$, $a^2 = (-a)^2$, $-b > 0$, and $-a > 0$, it follows from 4 (a) that $b^2 > a^2 \rightarrow -b > -a \rightarrow a > b$.

5. Prove: For any real number a and any positive number b,

(a) $a^2 \geq b$ if and only if $a \geq \sqrt{b}$ or $a \leq -\sqrt{b}$.

Suppose $a^2 \geq b$. Now, if $a^2 = b$, then $a = \sqrt{b}$, or $a = -\sqrt{b}$; hence $a \geq \sqrt{b}$ or $a \leq -\sqrt{b}$. So, assume $a^2 > b$. If $a > 0$ then $a^2 > (\sqrt{b})^2$ implies that $a > \sqrt{b}$ by 4(a). $a = 0$ is impossible, since b was assumed positive. If $a < 0$, then $-a > 0$. So, again by 4(a), $-a > \sqrt{b}$ since $(-a)^2 = a^2 > (\sqrt{b})^2$. But by Problem 3, if $-a > \sqrt{b}$, then $(-a)(-1) < \sqrt{b}(-1)$ or $a < -\sqrt{b}$. Thus we see that if $a^2 > b$, then either $a > \sqrt{b}$ or $a < -\sqrt{b}$. Conversely, suppose $a \geq \sqrt{b}$ or $a \leq -\sqrt{b}$. If $a > \sqrt{b}$, then $a^2 > b$ from 4(a). If $a < -\sqrt{b}$, then, since $-\sqrt{b} < 0$, $a < 0$. So 4(b) applies and $a^2 > (-\sqrt{b})^2 = b$. Finally, if $a = \sqrt{b}$, then $a^2 = (\sqrt{b})^2 = b$, or if $a = -\sqrt{b}$, then $a^2 = (-\sqrt{b})^2 = b$. Thus, if $a \geq \sqrt{b}$ or $a \leq -\sqrt{b}$, then $a^2 \geq b$.

(b) $a^2 \leq b$ if and only if $-\sqrt{b} \leq a \leq \sqrt{b}$.

Suppose $a^2 \leq b$. If $a^2 = b$, then $a = \sqrt{b}$ or $a = -\sqrt{b}$, hence $-\sqrt{b} \leq a \leq \sqrt{b}$. So, suppose $a^2 < b$. Again, if $a > 0$, then 4(a) applies, and since $a^2 < (\sqrt{b})^2$, $a < \sqrt{b}$. But, $-\sqrt{b} < 0 \leq a \leq \sqrt{b}$, so $-\sqrt{b} \leq a \leq \sqrt{b}$. If $a = 0$, then $-\sqrt{b} < 0 < \sqrt{b}$ since $\sqrt{b}$ positive, so again $-\sqrt{b} \leq a \leq \sqrt{b}$. Finally, if $a < 0$, then $(a)^2 < (-\sqrt{b})^2$ implies that $-\sqrt{b} < a$ from 4(b). So, $-\sqrt{b} < a < 0 < \sqrt{b}$, so $-\sqrt{b} \leq a \leq \sqrt{b}$. Thus, in all cases, $a^2 \leq b$ implies that $-\sqrt{b} \leq a \leq \sqrt{b}$. The proof of the converse proceeds similarly.

6. Prove that if $a > b > 0$ and $c > d > 0$, then $ac > bd$.

$$a > b \text{ and } c > 0 \longrightarrow ac > bc \ ;$$
$$b > 0 \text{ and } c > d \longrightarrow bc > bd \ .$$

Hence, $ac > bd$, by transitivity.

7. Prove: For any real numbers a and b if $ab > 0$, then either both $a > 0$ and $b > 0$ or both $a < 0$ and $b < 0$.

By trichotomy either: $a > 0$ $a = 0$ or $a < 0$
and if $b < 0$ or $b > 0$
then $ab < 0$ $ab = 0$ or $ab < 0$.

Contradiction.

So, if $a > 0$, then $b > 0$, or if $a < 0$, then $b < 0$.

8. Prove: For any real number a ,
(a) if $a > 0$, then $\frac{1}{a} > 0$;
(b) If $a < 0$, then $\frac{1}{a} < 0$.

$1 > 0$, and $a > 0$ and $a \cdot \frac{1}{a} = 1$. So, $a \cdot \frac{1}{a} > 0$. From Number 7, parts 8(a) and 8(b), follow. That is, if $a > 0$ then $\frac{1}{a} > 0$ and if $a < 0$ then $\frac{1}{a} < 0$.

9. For $bd < 0$, show that $\frac{a}{b} < \frac{c}{d}$ if and only if $ad > bc$.

If $bd < 0$ and $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b}(bd) > \frac{c}{d}(bd)$ or $ad > bc$.

If $bd < 0$ and $ad > bc$, then $\frac{1}{bd} < 0$.

Hence, $ad(\frac{1}{bd}) < bc(\frac{1}{bd})$ or $\frac{a}{b} < \frac{c}{d}$.

10. Show that for two positive numbers a and b , if $a > b$, then $\frac{1}{a} < \frac{1}{b}$.

If $a > b > 0$, from No. 8 it follows that $\frac{1}{a} > 0$, $\frac{1}{b} > 0$, hence that $\frac{1}{ab} > 0$. Thus $a(\frac{1}{ab}) > b(\frac{1}{ab})$ or $\frac{1}{b} > \frac{1}{a}$.

11. Prove that the complex numbers form a field $\mathcal{C}$ and that there can be no order relation on $\mathcal{C}$.

The proof that the complex numbers form a field is quite routine, with the possible exception of the existence of the multiplicative inverse. This proceeds as follows. Let $a + bi \neq 0$. We wish to find a complex z, so that $(a + bi)z = 1$. As motivation, we could write $z = \frac{1}{a + bi}$, and then rationalize, $(\frac{1}{a + bi})(\frac{a - bi}{a - bi}) = \frac{a - bi}{a^2 + b^2}$. Since $a + bi \neq 0$, $a \neq 0$ or $b \neq 0$, so $a^2 + b^2 \neq 0$. Thus, $z = \frac{a}{a^2 + b^2} - \frac{bi}{a^2 + b^2}$ is well defined. So, we claim that $(a + bi)(\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i) = 1$.

We obtain
$$\frac{a^2}{a^2 + b^2} - \frac{b^2i^2}{a^2 + b^2} + (\frac{ab}{a^2 + b^2} - \frac{ab}{a^2 + b^2})i = \frac{a^2 - b^2(-1)}{a^2 + b^2}$$
$$= \frac{a^2 + b^2}{a^2 + b^2} = 1 .$$
Now, notice that all the facts that have been proved for the ordering on the real numbers would hold for any ordered field. This is because all our proofs were based on the field and order axioms alone, not on any other properties of the real numbers. Thus, in any ordered field, $a^2 > 0$ all $a \neq 0$. So, if the complex numbers were ordered by $>$, then $i^2 > 0$, since $i \neq 0$. $i^2 = -1$, so $-1 > 0$. But $1^2 = 1 > 0$ also. Hence, $1 + (-1) > 0 + (-1)$ by the addition law. So, $0 > -1$. But, we already deduced that $-1 > 0$. Thus the trichotomy law is contradicted. Thus, there can be no order relation on the complex numbers.

12. The field $\mathcal{F}$ of numbers of the form $a + b\sqrt{2}$ where a and b are rational numbers has the ordering relation $>$ because $\mathcal{F}$ is a subset of $\mathcal{R}$. Show that $\mathcal{F}$ is also ordered by the relation $\succ$ where
$$a + b\sqrt{2} \succ c + d\sqrt{2} \quad \text{means that}$$
$$a - b\sqrt{2} > c - d\sqrt{2} .$$

First, we note that if $a + b\sqrt{2} = c + d\sqrt{2}$, then $a = c$ and $b = d$. For if $a + b\sqrt{2} = c + d\sqrt{2}$, then $a - c = (d - b)\sqrt{2}$. If $d \neq b$, then $\frac{a - c}{d - b} = \sqrt{2}$. This is impossible, since $\sqrt{2}$ is irrational, and $\frac{a - c}{d - b}$ is rational. So, $d = b$. Then $a = c$. Now, $\succ$ satisfies the trichotomy law, since $>$ satisfies the trichotomy law for the real numbers. We have $a - b\sqrt{2} > c - d\sqrt{2}$, $a - b\sqrt{2} < c - d\sqrt{2}$, or $a - b\sqrt{2} = c - d\sqrt{2}$. In the first two cases we have, by definition, that $a + b\sqrt{2} \succ c + d\sqrt{2}$, or $a + b\sqrt{2} \prec c + d\sqrt{2}$. But if $a - b\sqrt{2} = c - d\sqrt{2}$, then by our remark at the beginning $a = c$ and $b = d$, so $a + b\sqrt{2} = c + d\sqrt{2}$. Thus, $a + b\sqrt{2} \succ c + d\sqrt{2}$, $a + b\sqrt{2} = c + d\sqrt{2}$, or $a + b\sqrt{2} \prec c + d\sqrt{2}$. But, $a + b\sqrt{2}$ and $c + d\sqrt{2}$ can satisfy at most one of these relations. For, the first and last relation cannot both be satisfied, since then by definition $a - b\sqrt{2} > c - d\sqrt{2}$ and $a - b\sqrt{2} < c - d\sqrt{2}$, and this would contradict the fact that $>$ satisfies trichotomy on the field of reals. Finally, if $a + b\sqrt{2} = c + d\sqrt{2}$, $a = c$ and $b = d$, then $a - b\sqrt{2} = c - d\sqrt{2}$, and again this is the only relation possible since $>$ satisfies trichotomy.

Transitive Law. This follows immediately from the fact that $>$ satisfies transitive law.

Addition Law. Suppose $a + b\sqrt{2} \succ c + d\sqrt{2}$. Let $a_1 + b_1\sqrt{2}$ be a third element of the field. Then, $a - b\sqrt{2} > c - d\sqrt{2}$ by definition, so $a - b\sqrt{2} + (a_1 - b_1\sqrt{2}) > c - d\sqrt{2} + (a_1 - b_1\sqrt{2})$ by the addition law for $>$. But then $(a + a_1) - (b + b_1)\sqrt{2} > (c + a_1) - (d + b_1)\sqrt{2}$. So, $(a + a_1) + (b + b_1)\sqrt{2} \succ (c + a_1) + (d + b_1)\sqrt{2}$, by definition of $\succ$. Again, $(a + b\sqrt{2}) + (a_1 + b_1\sqrt{2}) \succ (c + d\sqrt{2}) + (a_1 + b_1\sqrt{2})$. Thus, letting $\alpha = a + b\sqrt{2}$, $\beta = c + d\sqrt{2}$, $\gamma = a_1 + b_1\sqrt{2}$, we have shown that for any α , β , and γ in $\mathcal{F}$, if $\alpha \succ \beta$, then $\alpha + \gamma \succ \beta + \gamma$. So the addition law is satisfied.

Multiplication Law. Let $\alpha = a + b\sqrt{2}$, $\beta = c + d\sqrt{2}$, $\gamma = q + s\sqrt{2}$. We assume $\alpha \succ \beta$ and $\gamma \succ 0$ and must prove $\alpha\gamma \succ \beta\gamma$. Then $a - b\sqrt{2} > c - d\sqrt{2}$ by definition. Since $\gamma \succ 0$, we have $q - s\sqrt{2} > 0$. By the multiplication law for $>$,

$$(a - b\sqrt{2})(q - s\sqrt{2}) > (c - d\sqrt{2})(q - s\sqrt{2}) .$$

But $(a - b\sqrt{2})(q - s\sqrt{2}) = (aq + 2bs) - (as + bq)\sqrt{2}$

and $(c - d\sqrt{2})(q - s\sqrt{2}) = (cq + 2ds) - (cs + dq)\sqrt{2}$.

Thus, $(aq + 2bs) - (as + bq)\sqrt{2} > (cq + 2ds) - (cs + dq)\sqrt{2}$.

So, $(aq + 2bs) + (as + bq\sqrt{2}) \succ (cq + 2ds) + (cs + dq)\sqrt{2}$ by definition.

But $(aq + 2bs) + (as + bq)\sqrt{2} = (a + b\sqrt{2})(q + s\sqrt{2})$

and $$(cq + 2ds) + (cs + dq)\sqrt{2} = (c + d\sqrt{2})(q + s\sqrt{2})\ .$$ Thus we have

$$(a + b\sqrt{2})(q + s\sqrt{2}) > (c + d\sqrt{2})(q + s\sqrt{2})$$

or

$$\alpha\gamma > \beta\gamma\ .$$

13. Show that for all real numbers x and y, $x^2 + xy + y^2 \geq 0$.

$$x^2 + xy + y^2 = (x + \frac{y}{2})^2 + (\frac{y\sqrt{3}}{2})^2 \geq 0$$

Alternatively,

$$x^2 + xy + y^2 \geq x^2 - 2|xy| + y^2 = (|x| - |y|)^2 \geq 0\ .$$

14. (a) Prove that $(x + y)^2 \geq 4xy$

$$(x + y)^2 - 4xy = (x - y)^2 \geq 0$$

(b) For positive numbers a and b, show that the arithmetic mean is not less than the geometric mean which is, in turn, greater than or equal to the harmonic mean:

$$\frac{a + b}{2} \geq \sqrt{ab} \geq \frac{2ab}{a + b}\ .$$

When does equality hold in this relation?

Since the 3 numbers in the inequality are positive, it suffices to show that

$$(\frac{a + b}{2})^2 \geq ab \geq \frac{4a^2b^2}{(a + b)^2}$$

By part (a), $(\frac{a + b}{2})^2 \geq ab$ and this implies $1 \geq \frac{4ab}{(a + b)^2}$

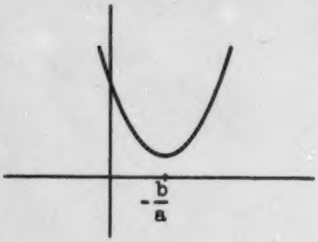
which proves the second part on multiplying by ab. The sign of equality holds if and only if $a = b$.

15. Find all values of x for which

$$ax^2 + 2bx + c \geq 0\ , \qquad a \neq 0\ .$$

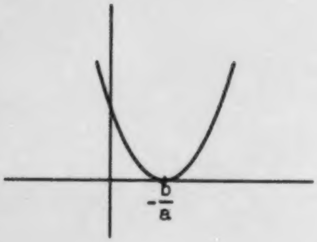
Discuss all possible cases.

Let $f(x) = ax^2 + 2bx + c$. From Intermediate Mathematics, it is known that the graph of f is a parabola which crosses the x-axis at $x = \frac{-b \pm \sqrt{b^2 - ac}}{a}$ (if the roots are real) and has its highest or lowest point at $x = -\frac{b}{a}$. There are 6 cases corresponding to $a > 0$, $a < 0$, and $b^2 - ac < 0$, $b^2 - ac = 0$, and $b^2 - ac > 0$. Consider the 3 cases for $a > 0$.



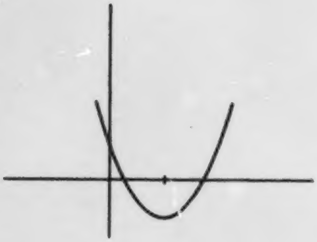
$b^2 - ac < 0$

Here, $ax^2 + 2bx + c > 0$ for all x.



$b^2 - ac = 0$

Here, $ax^2 + 2bx + c \geq 0$ for all x.



$b^2 - ac > 0$

Here, $ax^2 + 2bx + c \geq 0$ for

$$x \geq \frac{-b + \sqrt{b^2 - ac}}{a} \quad \text{and} \quad x \leq \frac{-b - \sqrt{b^2 - ac}}{a} .$$

For $a < 0$, the parabola opens downward and we obtain slightly different results.

Alternatively,

$$ax^2 + 2bx + c = a \left[\left(x + \frac{b}{a}\right)^2 + \frac{ac - b^2}{a^2} \right] .$$

The results for each of the six cases follow directly from this form.

16. Observe that

$$(a_1x + b_1)^2 + (a_2x + b_2)^2 + \dots + (a_nx + b_n)^2 \geq 0 ;$$

then use the solution of Number 15, to prove the Cauchy Inequality:

$$(a_1b_1 + a_2b_2 + \dots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2) ,$$

with equality, if and only if $a_r = kb_r$, or $b_r = 0$, for $r = 1 , 2 , \dots , n$ and k , some constant.

Since $(a_1x + b_1)^2 \geq 0$, for all values of x , then

$$(a_1x + b_1)^2 + (a_2x + b_2)^2 + \dots + (a_nx + b_n)^2 \geq 0$$

for all values of x .

Simplifying:

$$(a_1^2 + a_2^2 + \dots + a_n^2)x^2 + 2(a_1b_1 + a_2b_2 + \dots + a_nb_n)x$$
$$+ (b_1^2 + b_2^2 + \dots + b_n^2) \geq 0 .$$

Now using Number 15 with

$$a = a_1^2 + a_2^2 + \dots + a_n^2$$

$$b = a_1b_1 + a_2b_2 + \dots + a_nb_n$$

and $c = b_1^2 + b_2^2 + \dots + b_n^2$, the desired inequality follows from the condition that $b^2 - ac \leq 0$. To have equality,

$$(a_ix + b_i)^2 = 0 , i = 1 , 2 , \dots , n$$

or $a_r = kb_r$ (k constant).

17. If $a_1, a_2, \ldots, a_n$ are positive numbers, show that

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}} .$$

(Arithmetic Mean) $\geq$ (Harmonic Mean)

This generalizes part of 14(b).

By Cauchy's Inequality,

$$[(\sqrt{a_1})^2 + (\sqrt{a_2})^2 + \ldots + (\sqrt{a_n})^2]\left[\frac{1}{(\sqrt{a_1})^2} + \frac{1}{(\sqrt{a_2})^2} + \ldots + \frac{1}{(\sqrt{a_n})^2}\right] \geq n^2 .$$

Equivalently,

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}} .$$

18. Prove the general triangle inequality:

$$\sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} + \sqrt{y_1^2 + y_2^2 + \ldots + y_n^2}$$
$$\geq \sqrt{(x_1-y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2} .$$

Squaring both sides and simplifying, we now need to prove

$$2\sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}\ \sqrt{y_1^2 + y_2^2 + \ldots + y_n^2}$$
$$\geq -2(x_1y_1 + x_2y_2 + \ldots + x_ny_n) .$$

The preceding inequality follows directly from Cauchy Inequality.

TC A1-3. Absolute Value and Inequality.

In Section A2-1 (footnote), we define $|a|$ as $\sqrt{a^2}$. This definition has the virtue of emphasizing the positivity of the square root. It also helps to prevent the error of writing $\sqrt{a^2} = a$ in case $a < 0$. This error leads to the amusing "proof":

$$1 = \sqrt{1} = \sqrt{(-1)^2} = -1$$

thus

$$1 = -1 .$$

We note that the form $\sqrt{a^2}$ lends itself, more conveniently, to mathematical manipulation (e.g., in Exercises 5-2b, No. 11, the expression for $f'(x)$ requires careful inspection).

Solutions Exercises A1-3

1. Find the absolute value of the following numbers.

(a)	-1.75	1.75
(b)	$-\frac{\pi}{4}$	$\frac{\pi}{4}$
(c)	$\sin(-\frac{\pi}{4})$	$\frac{\sqrt{2}}{2}$
(d)	$\cos(-\frac{\pi}{2})$	0

2. (a) For what real numbers x does $\sqrt{x^2} = -x$?

   $x \leq 0$

   (b) For what real numbers x does $|1 - x| = x - 1$?

   $x \geq 1$

3. **Solve the equations:**

(a) $|3 - x| = 1$	$x = 2$ or $x = 4$.
(b) $|4x + 3| = 1$	$x = -\frac{1}{2}$ or $x = -1$.
(c) $|x + 2| = x$	Either $x + 2 \geq 0$ or $x + 2 < 0$, then $x + 2 = x$ or $-(x + 2) = x$. Thus there are no solutions.
(d) $|x + 1| = |x - 3|$	The only solution is $x = 1$.
(e) $|2x + 5| + |5x + 2| = 0$	There are no solutions.
(f) $|2x + 3| = |5 - x|$	$x = -8$ or $x = \frac{2}{3}$.
(g) $2|3x + 4| + |x - 2| = 1 + |3 + x|$	There are no solutions.

4. For what values of x is each of the following true? (Express your answer in terms of inequalities satisfied by x .)

(a) $|x| \leq 0$	$x = 0$
(b) $|x| \neq x$	$x < 0$
(c) $|x| < 3$	$-3 < x < 3$
(d) $|x - 6| \leq 1$	$5 \leq x \leq 7$
(e) $|x - 3| > 2$	$x < 1$ or $x > 5$
(f) $|2x - 3| < 1$	$1 < x < 2$
(g) $|x - a| < a$	$0 < x < 2a$
(h) $|x^2 - 3| < 1$	$\sqrt{2} < x < 2$ or $-2 < x < -\sqrt{2}$
(i) $|(x - 2)(x - 3)| > 2$	$x < 1$ or $x > 4$
(j) $|x - 1| > |x - 3|$	$x > 2$
(k) $|x - 5| + 1 = |x + 5|$	$x = \frac{1}{2}$

(l)	$\|x - 1\| + \|x - 2\| = 1$	$1 \le x \le 2$
(m)	$\|x^2 - a^2\| > 0$	$x \neq \pm a$
(n)	$\|x - a\| < \delta$	$a - \delta < x < a + \delta$
(o)	$0 < \|x - a\| < \delta$	$a - \delta < x < a$ or $a < x < a + \delta$
(p)	$\|x - 1\| < 2$ and $\|x + 1\| < \frac{3}{2}$	$-1 < x < \frac{1}{2}$
(q)	$\|x - 1\| < 2$ and $\|2x - 1\| < \frac{3}{2}$	$-\frac{1}{4} < x < \frac{5}{4}$
(r)	$\|x + y\| = \|x\| + \|y\|$, for all y	$x = 0$
(s)	$\|\sin x\| = 0$	$x = n\pi$, n, an integer
(t)	$\|\sin x\| > \frac{\sqrt{2}}{2}$	$\frac{\pi}{4} + \pi n < x < \frac{3\pi}{4} + \pi n$
(u)	$\|1 - \frac{1}{x}\| < 1$	$x > \frac{1}{2}$
(v)	$\sqrt{\|x\|} > \frac{1}{2}$	$x < -\frac{1}{4}$ or $x > \frac{1}{4}$

5. Sketch the graphs of the following equations:

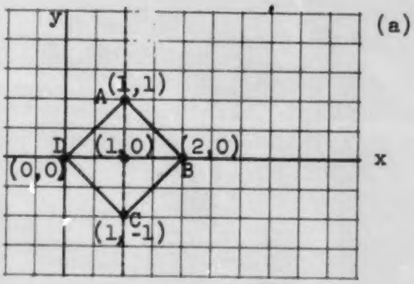
(a) $|x - 1| + |y| = 1$

For $x \ge 1$, $y > 0$, then
$x - 1 + y = 1$ or $x + y = 2$,
line AB .

For $x \ge 1$, $y < 0$, then
$x - 1 - y = 1$ or $x - y = 2$,
line BC .

For $x < 1$, $y < 0$, then
$-x + 1 - y = 1$ or
$x + y = 0$, line CD .

For $x < 1$, $y \ge 0$, then
$-x + 1 + y = 1$, or
$y = x$, line DA .

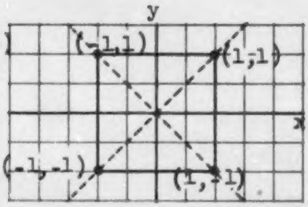


(b) $|x + y| + |x - y| = 2$

Resolves into 4 parts:

$x = \pm 1$ and $y = \pm 1$
where $|x| \leq 1$ and $|y| \leq 1$

(b)



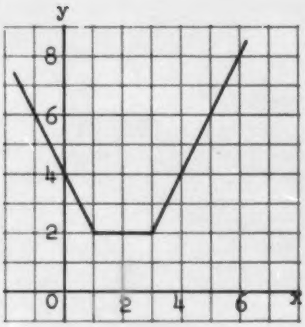
(c) $y = |x - 1| + |x - 3|$

For $x < 1$, then $y = -2x + 4$.

For $1 \leq x \leq w$, then $y = 2$.

For $x > 3$, then $y = 2x - 4$.

(c)



(d) $y = |x - 1| + |x - 3| + 2|x - 4|$

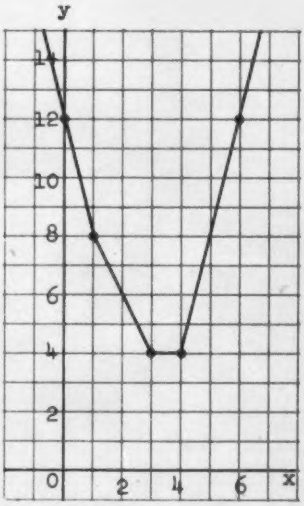
For $x < 1$, then $y = -4x + 12$.

For $1 \leq x < 3$, then $y = -2x + 10$.

For $3 \leq x < 4$, then $y = 4$.

For $4 \leq x$, then $y = 4x - 12$.

(d)



(e) $y = |x - 1| + |x - 3| + 2|x - 4| + 3|x - 5|$

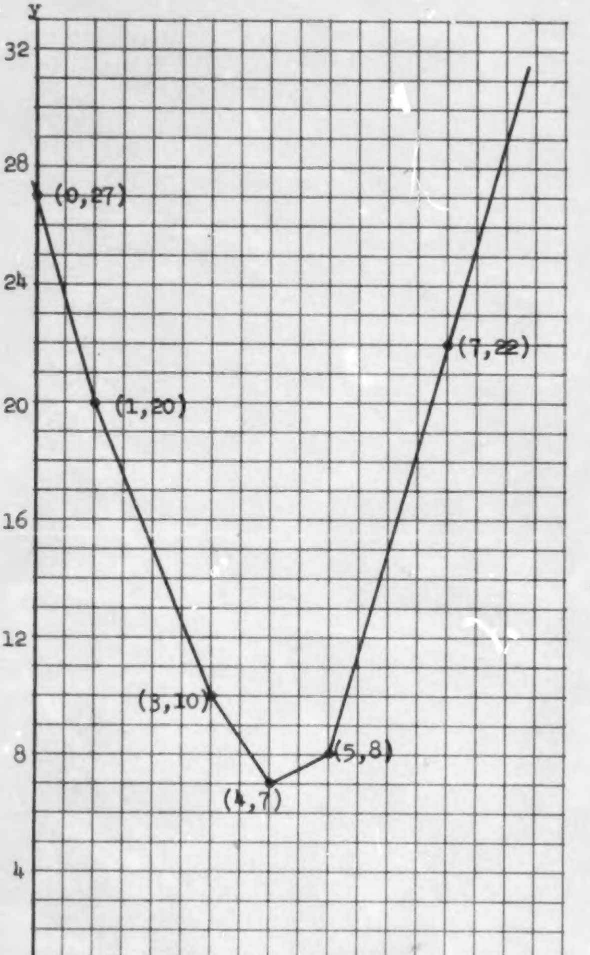
For $x < 1$, then $y = -7x + 27$.

For $1 \leq x < 3$, then $y = -5x + 25$.

For $3 \leq x < 4$, then $y = -3x + 19$.

For $4 \leq x < 5$, then $y = x + 3$.

For $5 \leq x$, then $y = 7x - 27$.



6. (a) Show that if $a > b > 0$, then $\frac{ab}{a+b} < b$.

$$0 < b \Longrightarrow a < a + b$$
$$\Longrightarrow ab < b(a + b)$$
$$\Longrightarrow \frac{ab}{a+b} < b \text{ , since } a + b > 0 .$$

(b) Thus, show that for positive numbers a and b , the condition $\delta \leq \min(a,b)$ is satisfied by $\delta = \frac{ab}{a+b}$.

For $a \neq b$ the result follows from part (a). For $a = b$, $\delta = \frac{a}{2} < a$.

7. (a) Show for positive a , b that $\frac{a+b}{2} < \max(a,b)$ if $a \neq b$.

$$\frac{a+b}{2} < \frac{\max(a,b) + \max(a,b)}{2} \leq \max(a,b)$$

(b) Prove for all a,b that

$$\max(a,b) = \frac{1}{2}(a + b + |a - b|)$$

(c) $$\min(a,b) = \frac{1}{2}(a + b - |a - b|)$$

Assume, without loss of generality, that $a \geq b$,

then $\max(a,b) = a = \frac{1}{2}(a + b + a - b)$,

and $\min(a,b) = b = \frac{1}{2}(a + b - (a-b))$.

8. Show that

$$\max(a,b) + \max(c,d) \geq \max(a + c , b + d) .$$

From Number 15

$$\max(a,b) + \max(c,d) = \frac{1}{2}(a + b + c + d + |a - b| + |c - d|) ,$$
$$\max(a + c , b + d) = \frac{1}{2}(a + b + c + d + |a + c - (b + d)|)$$
$$= \frac{1}{2}(a + b + c + d + |(a - b) + (c - d)| .$$

The result follows at once.

9. Show that if $ab \geq 0$, then $ab \geq \min(a^2,b^2)$.

$$ab = |a|\ |b| \geq (\min(|a|,|b|))^2 = \min(a^2,b^2) .$$

10. Show that if $a = \max(a,b,c)$, then $-a = \min(-a,-b,-c)$.

If $\qquad a = \max(a,b,c)$, $a \geq b$, $a \geq c$,

then $\qquad -a \leq -b$ and $-a \leq -c$.

So, $\qquad -a = \min(-a,-b,-c)$.

11. Denote $\min\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}\right\}$ by $\min\limits_r\left\{\frac{a_r}{b_r}\right\}$ and similarly for max.

If $b_r > 0$, $r = 1, 2, \ldots, n$, prove that

$$\min_r\left\{\frac{a_r}{b_r}\right\} \leq \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} \leq \max_r\left\{\frac{a_r}{b_r}\right\} .$$

Denote $\min\limits_r\left\{\frac{a_r}{b_r}\right\}$ by $\frac{a_k}{b_k}$, $1 \leq k \leq n$, and

$\max\limits_r\left\{\frac{a_r}{b_r}\right\}$ by $\frac{a_e}{b_e}$, $1 \leq e \leq n$.

Then, $\frac{a_k}{b_k} \leq \frac{a_r}{b_r}$, $r = 1, 2, \ldots, n$.

Or, $\quad a_kb_r \leq b_ka_r$, for all r. Adding,

$$a_kb_1 + a_kb_2 + \ldots + a_kb_n \leq b_ka_1 + b_ka_2 + \ldots + b_ka_n .$$

Factoring and dividing, we obtain

$$\min\left\{\frac{a_r}{b_r}\right\} = \frac{a_k}{b_k} \leq \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + a_n} .$$

In the same way the remaining inequality may be obtained.

12. Prove that

$$\frac{1}{n} \leq \frac{1+2+\dots+n}{(n)^2+(n-1)^2+\dots+2^2+1^2} \leq 1 \quad \text{for} \quad n = 1, 2, 3, \dots, n .$$

Use the inequality obtained in Number 11, with

$$\frac{a_r}{b_r} = \frac{r}{r^2} \quad \text{or} \quad \frac{1}{r} .$$

13. (a) Prove directly from the properties of order for $\epsilon > 0$ that if $-\epsilon \leq x \leq \epsilon$ then $|x| \leq \epsilon$. Conversely, if $|x| \leq \epsilon$ then $-\epsilon \leq x \leq \epsilon$.

Suppose $-\epsilon \leq x \leq \epsilon$. If $0 < x$, $|x| = x \leq \epsilon$. If $x < 0$, $|x| = -x$. But $-\epsilon \leq x$ implies $-x \leq \epsilon$. So, $-x = |x| \leq \epsilon$.

Conversely, suppose $|x| \leq \epsilon$. If $0 \leq x$, $|x| = x$, so $-\epsilon < 0 \leq x \leq \epsilon$, thus $-\epsilon \leq x \leq \epsilon$. Similarly for $x < 0$.

(b) Prove that if x is an element of an ordered field and if $|x| < \epsilon$ for all positive values ϵ, then $x = 0$.

If $x \neq 0$, take $\epsilon = |x|$. We then have the contradictory statements $|x| = |x|$ and $|x| < |x|$.

14. (a) Prove that $|ab| = |a| \cdot |b|$.

Just consider the three cases

$$ab > 0 , \quad ab = 0 , \quad ab < 0 .$$

(b) Prove that $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, $b \neq 0$.

From part(a) we have

$$\left|\frac{a}{b}\right| = \left|a\left(\frac{1}{b}\right)\right| = |a|\left|\frac{1}{b}\right|$$

and

$$|b| \cdot \left|\frac{1}{b}\right| = \left|\frac{b}{b}\right| = |1| = 1 .$$

Hence $\left|\frac{1}{b}\right| = \frac{1}{|b|}$.

Therefore $|a|\left|\frac{1}{b}\right| = |a| \cdot \frac{1}{|b|} = \frac{|a|}{|b|}$.

15. Prove that $|x - y| \leq |x| + |y|$.

In $|a + b| \leq |a| + |b|$, set $a = x$, $b = -y$.

16. Under what conditions do the equality signs hold for

$$\big||a| - |b|\big| \leq |a + b| \leq |a| + |b| \ ?$$

Equality occurs only if $a = b = 0$.

17. If $0 < x < 1$, we can multiply both sides of the inequality $x < 1$ by x to obtain $x^2 < x$ (and, similarly, we can show that $x^3 < x^2$, $x^4 < x^3$, and so on). Use this result to show that if $0 < |x| < 1$, then $|x^2 + 2x| < 3|x|$.

$$|x^2 + 2x| \leq |x^2| + |2x| \leq |x| + |2x| = 3|x| ,$$

$$(|x^2| = |x|^2 < |x| \text{ since } 0 < |x| < 1) .$$

18. Prove the following inequalities

(a) $x + \frac{1}{x} \geq 2$, $x > 0$.

Since $(x - 1)^2 \geq 0$, we have

$$x^2 - 2x + 1 \geq 0 \quad \text{or} \quad x^2 + 1 \geq 2x .$$

Since $\frac{1}{x} > 0$, we obtain $x + \frac{1}{x} \geq 2$.

(b) $x + \frac{1}{x} \leq -2$, $x < 0$.

$(x + 1)^2 \geq 0$. So $x^2 + 2x + 1 \geq 0$, or $x^2 + 1 \geq -2x$.

Since $x < 0$, $\frac{1}{x} < 0$, so $\frac{1}{x}(x^2 + 1) \leq \frac{1}{x}(-2x)$ or $x + \frac{1}{x} \leq -2$.

(c) $|x + \frac{1}{x}| \geq 2$, $x \neq 0$.

From (a) we have $x + \frac{1}{x} \geq 2$ for $x > 0$

or $|x + \frac{1}{x}| \geq 2$ for $x > 0$.

From (b) we have $-(x + \frac{1}{x}) \geq 2$ for $x < 0$

or $-(x + \frac{1}{x}) = |x + \frac{1}{x}| \geq 2$ for $x < 0$.

Thus $|x + \frac{1}{x}| \geq 2$ for $x \neq 0$.

19. Prove: $x^2 \geq x|x|$ for all real x.

If $x \geq 0$, $x = |x|$, and $x^2 = x|x|$.

If $x < 0$, $x|x| < 0 < x^2$.

20. Show that if $|x - a| < \frac{|a|}{2}$, then $\frac{|a|}{2} < |x| < \frac{3|a|}{2}$ for all $a \neq 0$.

Using the inequalities of 13(a), we obtain

$$|x - a| < \frac{|a|}{2} \longrightarrow -\frac{|a|}{2} < x - a < \frac{|a|}{2}$$

$$\longrightarrow \frac{|a|}{2} < x < \frac{3|a|}{2} .$$

21. Prove for positive a and b , where $a \neq b$, that

$$\frac{|b - a|^2}{4(a + b)} < \frac{a + b}{2} - \sqrt{ab} < \frac{|b - a|^2}{8\sqrt{ab}} .$$

To avoid $\sqrt{\ }$, let $a = m^2$, $b = n^2$, then we have to show that

$$\frac{|m^2 - n^2|^2}{4(m^2 + n^2)} < \frac{m^2 + n^2 - 2mn}{2} < \frac{|m^2 - n^2|^2}{8mn}$$

or

$$\frac{(m + n)^2}{2(m^2 + n^2)} < 1 < \frac{(m + n)^2}{4mn} .$$

Both of these inequalities are equivalent to $(m - n)^2 > 0$.
By way of example we show one of the equivalences:

$$\begin{aligned} (m - n)^2 > 0 &\longrightarrow m^2 + n^2 > 2mn \\ &\longrightarrow 2(m^2 + n^2) > (m + n)^2 \\ &\longrightarrow 1 > \frac{(m + n)^2}{2(m^2 + n^2)} . \end{aligned}$$

Solutions Exercises A1-4

1. Use absolute value and inequalities to express the following facts.
   (a) The point x is closer to -2 than is the point a .

   $|x + 2| < |a + 2|$

   (b) The point x is closer to point a then it is to the origin.

   $|x - a| < |x|$

2. In each case, use absolute values and inequalities to express the fact that x is in the interval.

(a) $[-5,-2]$	$\lvert x + \frac{7}{2}\rvert \leq \frac{3}{2}$
(b) $(-1,5)$	$\lvert x - 2\rvert < 3$
(c) $[5.9,6.1]$	$\lvert x - 6\rvert \leq .1$
(d) $(-2.95,-2.85)$	$\lvert x + 2.9\rvert < .05$

3. Find the interval or deleted interval to which all values of x must belong for each of the following:

(a)	$\|x + 2\| < 1$	$(-3,-1)$
(b)	$0 < \|x + 2\| < 1$	$(-3,-1)$ with -2 deleted.
(c)	$\|x + a\| < \frac{\|a\|}{2}$	$(-a - \frac{\|a\|}{2}, -a + \frac{\|a\|}{2})$
(d)	$0 < \|x + a\| < \frac{\|a\|}{2}$	$(-a - \frac{\|a\|}{2}, -a + \frac{\|a\|}{2})$ with $-a$ deleted.

4. (a) A set of points is said to be bounded if there exists a real number A such that $|x| < A$, for all members x of the set. Which of the intervals in Number 3 are bounded? Which are not? Prove your assertions.

They are all bounded.

(a) and (b). Since $|x| - 2 \leq |x + 2| < 1$, then $|x| < 3$ if x lies in the set considered.

(c) and (d). Again, since $|x| - |a| \leq |x + a| < \frac{|a|}{2}$, then $|x| < \frac{3|a|}{2}$.

(b) A number M is said to be an upper bound for the set if $x \leq M$ for all members of the set; a number m is said to be a lower bound if $x \geq m$ for all members of the set. If a set has an upper bound, is it necessarily bounded? What if it has both an upper and a lower bound? If a set is bounded does it have both an upper and a lower bound? Prove your assertions.

No; the set S consisting of all $x < 0$ is bounded above by 0, but is not bounded.

Yes; if a set S has an upper and a lower bound, it is bounded. For, suppose $m \leq x \leq M$ for all x in the set. Let $L = 2\max\{|m|,|M|\}$. Then, $-L < -|m| \leq m \leq x \leq M \leq |M| < L$ or $-L < x < L$ or $|x| < L$ for all x in the set.

Yes; if the set is bounded, then there is an M so that $|x| < M$ for all x in the set. Then $|x| \leq M$, so $-M \leq x \leq M$, so M is an upper bound and $-M$ is a lower bound.

5. For each of the following statements give the interval or intervals on which the statement is true.

(a) $x^2 - x - 6 > 0$

or $(x - 3)(x + 2) > 0$. Thus either, $x - 3$ and $x + 2$ must both be greater, or both less, than zero, i.e., $x > 3$, or $x < -2$.

(b) $(x - a)(x - b)(x - c) \leq 0$, for $a < b < c$.

Three possible ways for this to happen.

(1) all 3 factors are negative.

(2) Only 1 factor is negative.

(3) One factor is zero.

So, either $x \leq a$ or $b \leq x \leq c$.

(c) $\cos x > \sin x$

An inspection of the graphs of $x \longrightarrow \sin x$ and $x \longrightarrow \cos x$ on the same axes will verify the same conclusion as reasoning that $\cos x > \sin x \Longrightarrow \tan x < 1$. Thus $-\frac{3}{4}\pi + 2\pi n < x < \frac{\pi}{4} + 2\pi n$ $(n = 0, \pm 1, \pm 2, \ldots)$.

(d) $x + 1 \geq 2\sqrt{x}$

or $x - 2\sqrt{x} + 1 \geq 0$ or $(\sqrt{x} - 1)^2 \geq 0$, which is true for all real numbers $(\sqrt{x} - 1)$. Thus the only restriction is that $x \geq 0$.

(e) $|x^2 - 1| < \frac{1}{100}$

Thus $1 - \frac{1}{100} < x^2 < 1 + \frac{1}{100}$. Whence, $\frac{\sqrt{99}}{10} < |x| < \frac{\sqrt{101}}{10}$.

6. In each of the following, for the given value of a find a neighborhood of a where the given inequality holds

(a) $a = \frac{3}{2}$, $|2x - 3| < \frac{1}{7}$. $\quad |x - \frac{3}{2}| < \frac{1}{14}$.

(b) $a = \frac{\pi}{2}$, $|\sin x - 1| < 1 - \frac{1}{\sqrt{2}}$.

$$|\sin x - 1| < 1 - \frac{1}{\sqrt{2}} \longrightarrow -1 + \frac{1}{\sqrt{2}} < \sin x - 1 < 1 - \frac{1}{\sqrt{2}}$$

$$\longrightarrow \frac{1}{\sqrt{2}} < \sin x < 2 - \frac{1}{\sqrt{2}}.$$

We then have $\frac{\pi}{4} < x < \frac{3\pi}{4}$ or $\frac{\pi}{2} - \frac{\pi}{4} < x < \frac{\pi}{2} + \frac{\pi}{4}$.

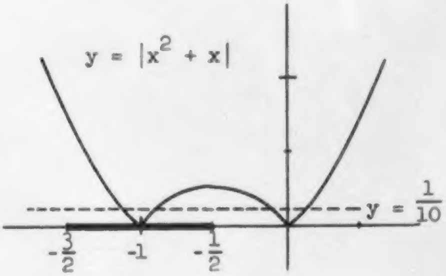
Thus $|x - \frac{\pi}{2}| < \frac{\pi}{4}$.

(c) $a = -1$, $|x^2 + x| < \frac{1}{10}$

(d) $a = -1$, $|x^2 + x| < \frac{1}{100}$

(e) $a = -1$, $|x^2 + x| < \frac{1}{1000}$

Consider the graph of $y = |x^2 + x|$ for (c), (d) and (e).



The graph suggests that we confine our discussion to the interval where $|x + 1| < \frac{1}{2}$; i.e., we shall require that $-\frac{3}{2} < x < -\frac{1}{2}$. Since

$$|x^2 + x| < \frac{1}{10} \Longrightarrow |x| \cdot |x + 1| < \frac{1}{10} \Longrightarrow |x + 1| < \frac{1}{10|x|}, \text{ and}$$

$\frac{1}{|x|} < \frac{2}{3}$; under these restrictions we have

(c) $|x + 1| < \frac{1}{10} \cdot \frac{2}{3} \leq \frac{2}{30}$.

(d) $|x + 1| < \frac{2}{300}$.

(e) $|x + 1| < \frac{2}{3000}$.

TC A1-5. Completeness of the Real Number System. The Separation Axiom.

The completeness of the real number system is a consequence of the Separation Axiom. We note that in the axiom the sets A and B might not be disjoint; e.g.,

$$A = \{x : x \leq 0\}\ ,\ B = \{x : x \geq 0\} \text{ and } 0$$

is the unique separation number. Observe that the word "separates" as used in the separation axiom does not mean that s is not in either A or B . The separation number s may be in either set, both sets, or neither set.

The completeness of the field of real numbers plays a central role in the rigorous development of the calculus. We have based our logical development upon the Separation Axiom because of its intuitive geometric content. It asserts the absence of gaps (holes) in the real number line. The Nested Interval Theorem, the Least Upper Bound Principle, and the Separation Axiom are all logically equivalent. Thus, in the sequel, we can feel free to use whichever is most convenient.

We have approached the real numbers axiomatically; i.e., we have given a set of axioms rich enough to yield the properties that we need. An alternative approach is to define the set of real numbers in terms of a more basic set, say the rational numbers. This approach was taken in the nineteenth century by Dedekind, Cantor, and others. The original article of Dedekind now appears in the paperback, Essays in the Theory of Numbers, by R. Dedekind, Dover, 1963. His account is both lucid and elementary.

Solutions Exercises A1-5

1. Prove Corollary 1 to the Least Upper Bound Principle. If M is the least upper bound of the set A , then for each positive ϵ there exists an $\alpha \in A$ such that $\alpha > M - \epsilon$.

   Suppose there is no $\alpha > M - \epsilon$. Then $M' : M - \epsilon < M' < M$ is an upper bound for A , contradicting M as the least upper bound.

2. Prove Corollary 2 to the Least Upper Bound Principle. A set of numbers which is bounded below has a greatest lower bound.

For the proof of Corollary 2, let $\bar{A}$ be a set which is bounded below and let $\bar{B}$ be the set of lower bounds of $\bar{A}$. The set $A = \{-\alpha : \alpha \in \bar{A}\}$ has as the set of its upper bounds, $B = \{-\beta : \beta \in \bar{B}\}$. The greatest lower bound $\bar{M}$ of $\bar{A}$ is given by $\bar{M} = -M$ where M is the least upper bound of A.

3. (a) Consider the sets A of positive rational numbers α satisfying $\alpha^2 < 2$, and B of positive rational numbers β satisfying $\beta^2 > 2$. Prove if $\alpha \in A$ and $\beta \in B$ that $\alpha < \beta$.

If $\alpha = \beta$, then $\alpha^2 = \beta^2$ contradicting $\alpha^2 < \beta^2$.
If $\alpha > \beta$, then since $\alpha > 0$ and $\beta > 0$, we have $\alpha^2 > \beta^2$.
These contradictions force the conclusion $\alpha < \beta$.

(b) Show that a separation number s for the sets A and B must satisfy $s^2 = 2$; i.e., $s = \sqrt{2}$.

Suppose s is a separation number for A and B and $s \neq \sqrt{2}$, say $s = \sqrt{2} + \epsilon$, $\epsilon > 0$. Then, $\sqrt{2} + \frac{\epsilon}{2} < s$ but $\sqrt{2} + \frac{\epsilon}{2} \in B$, since $(\sqrt{2} + \frac{\epsilon}{2})^2 = 2 + \epsilon\sqrt{2} + \frac{\epsilon^2}{4} > 2$. This contradiction indicates $s = \sqrt{2} + \epsilon$ is not a separation number. $s = \sqrt{2} - \epsilon$ can also be shown to fail, leaving only $s = \sqrt{2}$ as the separation number.

(c) Prove that $\sqrt{2}$ is irrational.

Assume $\frac{p}{q}$ is a rational solution in lowest terms of $x^2 = 2$.
Then, $p^2 = 2q^2$ implies p^2 is even and thus $p = 2r$, where r is an integer. Now $4r^2 = 2q^2$ implies q is even which contradicts our original assumption that $\frac{p}{q}$ is in lowest terms. Hence $\sqrt{2}$ is irrational.

4. (a) Prove for every real number a, that there is an integer n greater than a (Principle of Archimedes).

Suppose there were no integer greater than a. Then, since the integers would have an upper bound a, they would have a least upper bound M. The number $M - \frac{1}{2}$ would not be an upper bound since M is least. Consequently there is an integer $n > M - \frac{1}{2}$. Then $n + 1 > M + \frac{1}{2} > M$, contrary to the definitions of integers and least upper bound.

(b) Prove that given any $\epsilon > 0$ there is an integer n such that $0 < \frac{1}{n} < \epsilon$.

Part (a) proved that there is an integer $n > \frac{1}{\epsilon}$. Hence $\epsilon > \frac{1}{n} > 0$.

5. (a) We define the infinite decimal

$$c_0 . c_1 c_2 c_3 \cdots ,$$

where c_0 is an integer, and $c_1, c_2, c_3, \ldots$, are digits as the number r where

$$c_0 + \frac{c_1}{10} + \frac{c_2}{10^2} + \cdots + \frac{c_n}{10^n} \leq r < c_0 + \frac{c_1}{10} + \frac{c_2}{10^2} + \cdots + \frac{c_n + 1}{10^n} .$$

Show that the preceding inequality does, in fact, define a unique real number.

Set

$$a_n = c_0 + \frac{c_1}{10} + \cdots + \frac{c_n}{10^n}$$

$$b_n = c_0 + \frac{c_1}{10} + \cdots + \frac{c_n + 1}{10^n} .$$

We have $a_n \leq r < b_n$ for all n. Then r is a separation number for A, the set of all a_n, and B, the set of all b_n. Given $\epsilon > 0$, then $b_n - a_n = \frac{1}{10^n} < \epsilon$ for n sufficiently large. By Lemma A1-5, r is the unique separation number for A and B.

A(b) Given a real number r we define its decimal representation recursively in terms of the integer part function $[x]$ as follows:

$$c_0 = [r]$$

$$c_n = \left[10^n\left(r - c_0 - \frac{c_1}{10} - \frac{c_2}{10^2} - \dots - \frac{c_{n-1}}{10^{n-1}}\right)\right].$$

Show that the inequality in part (a) is satisfied for this choice of c_n. Show also that decimals consisting entirely of 9's from some point on are avoided. (Thus, we obtain $2 = 2.000\dots$ but not $2 = 1.999\dots$)

Since $x - 1 < [x] \le x$,

$$(1)\quad 10^n\left(r - c_0 - \dots - \frac{c_{n-1}}{10^{n-1}}\right) - 1 < c_n$$

$$\le 10^n\left(r - c_0 - \dots - \frac{c_{n-1}}{10^{n-1}}\right),$$

or equivalently

$$(2)\quad c_0 + \dots + \frac{c_n}{10^n} \le r < c_0 + \dots + \frac{c_n + 1}{10^n}.$$

(Note the strong inequality on the right.)

Next, we establish for $n \ge 1$ that c_n is a digit by mathematical indication.

For c_1, we have from (1) that $10(r - c_0) - 1 < c_1 \le 10(r - c_0)$ where $0 \le (r - c_0) < 1$.

Hence, $-1 < c_1 < 10$

or, since c_1 is an integer ≥ 0, $0 \le c_1 \le 9$. Now suppose that c_n is a digit $0 \le c_n \le 9$. In the notation of Number 5a, we see from (1) that

$$(3)\quad c_n \le 10^n(r - a_{n-1}) < 1 + c_n.$$

Now, replacing n by $n + 1$ in (1) we have

$$(4)\quad 10^{n+1}\left(r - a_{n-1} - \frac{c_n}{10^n}\right) - 1 < c_{n+1} \le 10^{n+1}\left(r - a_{n-1} - \frac{c_n}{10^n}\right).$$

Using the inequality (3) in (4) we obtain

$$-1 < c_{n+1} < 10$$

from which we conclude that c_{n+1} is a digit.

Now, let us suppose that r can be represented as a decimal with an infinite string of 9's

$$r = d_0 . d_1 d_2 \cdots d_{p-1}\, d_p\, 999 \cdots$$

where we may without loss of generality suppose that either $p = 0$ or $d_p \neq 9$ (i.e., that the last decimal place where a 9 does not appear, if there is any, is the p-th place). Consider the number

$$\rho = d_0 . d_1 d_2 \cdots (1 + d_p)\, 000 \cdots .$$

Take $d_q = 9$ for $q > p$. For any index q, then, we see that both ρ and r lie between the numbers

$$d_0 + \frac{d_1}{10} + \ldots + \frac{d_q}{10^q}$$

and
$$d_0 + \frac{d_1}{10} + \ldots + \frac{1 + d_q}{10^q} .$$

Since the upper and lower estimates here differ by $\frac{1}{10^q}$ it follows that $\rho = r$. Now the method of representation given above yields a unique decimal, since (2) is equivalent to the definition of d_n by means of the integer part function. Taking $c_q = d_q$ for $q < p$, $c_p = 1 + d_p$, and $c_q = 0$ for $q > p$ we see that equality holds on the left in (2) for $q \geq p$; hence any other representation then the terminating one is precluded.

6. An infinite decimal $c_0 . c_1 c_2 c_3 \ldots$ is said to be periodic if for some fixed value p, the period of the decimal, we have $c_{n+p} = c_n$ for all n satisfying $n \geq n_0$, where we require that p is the smallest positive integer satisfying this condition. In words, from some place on, the decimal consists of the indefinite repetition of the same p digits. Thus

$$\frac{1}{3} = .33333\cdots$$

$$\frac{15}{44} = .34090909\ldots$$

are periodic decimals. It is convenient to indicate a cycle of p digits by underlining, rather than repetition; e.g.,

$$\frac{22}{7} = 3.\underline{142857} .$$

(a) Prove that every periodic decimal represents a rational number. (Hint: Consider the decimal as a geometric progression.)

Let $\quad r = c_0 \,.\, c_1 \cdots\cdots c_q \underline{b_1 b_2 \cdots b_p}$

Then $\quad r = \frac{\gamma}{10^q} + \frac{\beta}{10^{p+q}} + \frac{\beta}{10^{2p+q}} + \frac{\beta}{10^{3p+q}} + \cdots$

where $\quad \gamma = 10^q \,(c_0 \,.\, c_1 c_2 \cdots c_q)$.

and $\quad \beta = 10^p \,(0 \,.\, b_1 b_2 \cdots b_p)$.

This latter representation for r is an infinite geometric series with common ratio $\frac{1}{10^p}$. Whence we have

$$r = \frac{\gamma}{10^q} + \frac{\beta}{10^{p+q}(1 - 10^{-p})}$$

$$= \frac{(10^p - 1)\gamma + \beta}{10^q(10^p - 1)},$$

which is a rational number, since $(10^p - 1)\gamma + \beta$ and $10^q(10^p - 1)$ are integers.

Λ(b) Prove that every number has a periodic decimal representation. (A "terminating" decimal in which each place beyond a certain point is zero is considered as a special case of periodic decimals.) If $r = \frac{s}{t}$ represents a rational number given in lowest terms, find the largest possible period of the infinite decimal representation of r in terms of the denominator t.

From (a) and (b) we conclude that a decimal which is not periodic represents an irrational number, and conversely.

We first submit a specific case which contains the germ of the proof: we discuss the rational number $\frac{8}{7}$.

$$\frac{8}{7} = 1 + \frac{1}{7}$$

$$\frac{1}{7} = \frac{1}{10}\left(\frac{10}{7}\right) = \frac{1}{10}\left(1 + \frac{3}{7}\right),$$

$$\frac{3}{7} = \frac{1}{10}\left(\frac{30}{7}\right) = \frac{1}{10}\left(4 + \frac{2}{7}\right),$$

$$\frac{2}{7} = \frac{1}{10}\left(\frac{20}{7}\right) = \frac{1}{10}\left(2 + \frac{6}{7}\right),$$

$$\frac{6}{7} = \frac{1}{10}\left(\frac{60}{7}\right) = \frac{1}{10}\left(8 + \frac{4}{7}\right),$$

$$\frac{4}{7} = \frac{1}{10}\left(\frac{40}{7}\right) = \frac{1}{10}\left(5 + \frac{5}{7}\right),$$

$$\frac{5}{7} = \frac{1}{10}\left(\frac{50}{7}\right) = \frac{1}{10}\left(7 + \frac{1}{7}\right).$$

Compare this procedure with the "long division" process:

$$
\begin{array}{r}
.142857 \\
7\,\overline{\big)1.000000} \\
\underline{7} \\
30 \\
\underline{28} \\
20 \\
\underline{14} \\
60 \\
\underline{56} \\
40 \\
\underline{35} \\
50 \\
\underline{49} \\
1\,.
\end{array}
$$

Note that we stop with a remainder 1 since it occured before. We would get a repetition of the same digits if we were to continue the division process:

$$\frac{8}{7} = 1.142857142857\ldots\ .$$

In general we shall always get a remainder which is repeated since there are only a finite numbers of different possible remainders. The argument given here is based on the division algorithm which must eventually repeat itself because remainders must recur.

Let $r = \frac{\sigma}{\tau}$ where σ and τ are relatively prime and $\tau > 0$. Let the decimal expansion of r as given by the method of Number 5 be

$$c_0\,.\,c_1c_2c_3\,\cdots\,.$$

Represent τ in the form 2^a5^bm where m has no factors of 2 or 5. Set $q = \max(a,b)$, $t = 10^qm$ and rewrite r in the form

$$r = \frac{s}{t} = \frac{s}{10^qm}\ .$$

Observe that m and s are relatively prime.
If $m = 1$ we have exactly

$$r = \frac{s}{10^q} = c_0\,.\,c_1c_2\,\cdots\,c_q$$

so that $c_n = 0$ for $n > q$ and the period is 1. If $m \neq 1$, take

$$\lambda_k = 10^{q+k}\,(c_0\,.\,c_1c_2\,\cdots\,c_{q+k})\ .$$

From (2) in the solution of Number 5, we have

$$\lambda_k \leq \frac{10^k s}{m} < \lambda_k + 1 ,$$

whence

$$0 \leq 10^k s - m\lambda_k < m .$$

Consequently, on dividing $10^k s$ by m we obtain the quotient λ_k and the remainder $r_k = 10^k s - m\lambda_k$. Now $r_k \neq 0$ since m and $10^k s$ are relatively prime. Thus r_k, as a remainder on division by m can only be one of the integers $1 , 2 , \ldots , m - 1$. It follows that at least two of the m numbers r_k, for $k = 1 , 2 , \ldots , m$, must be the same, say $r_i = r_j$ with $j > i$. From this we now prove that the decimal expansion for r is periodic with the period $p = j - i$, namely,

$$r = c_0 . c_1 c_2 \cdots c_{q+i} \underline{c_{q+i+1} \cdots c_{q+i+p}} .$$

We show first that $r_k = r_{k+p}$. Observe that,

$$r_{k+1} = 10^{k+1} s - m\lambda_{k+1} = 10^{k+1} s - m(10\lambda_k + c_{q+k+1})$$

$$= 10r_k - m\, c_{q+k+1} .$$

But $c_{q+k+1} = \left[10^{q+k+1} (r - c_0 . c_1 c_2 \cdots c_{q+k})\right]$

$$= \left[10\left(\frac{10^k s}{m} - \lambda_k\right)\right]$$

$$= \left[\frac{10r_k}{m}\right] .$$

Combining the last two results, we obtain a representation for r_{k+1} in terms of r_k alone. It follows from $r_i = r_j$ that $r_{i+1} = r_{j+1}$, $r_{i+2} = r_{j+2}$, etc. Thus r_k is periodic with period p. Since c_{q+k+1} is a function of r_k we see that the decimal also is periodic with period p.

Λ(c) **Prove for every positive prime α other than 2 and 5 that there exists an integer, all of whose digits are ones, for which α is a factor; i.e., α is a factor of some number of the form**

$$10^n + 10^{n-1} + 10^{n-2} + \dots + 10 + 1 .$$

Let α be the given prime. We can write (from part (a))

$$\frac{1}{\alpha} = \frac{(10^p - 1)\gamma + \beta}{10^q(10^p - 1)}$$

or

$$\alpha\Big((10^p - 1)\gamma + \beta\Big) = 10^q(10^p - 1) .$$

Since α is neither 2 nor 5 it follows that α is a factor of

$$10^p - 1 = 9(10^{p-1} + 10^{p-2} + \dots + 10 + 1) .$$

If $\alpha \neq 3$, then α must be a factor of the expression in parenthesis. If $\alpha = 3$, then α is a factor of $10^2 + 10 + 1$. In either case, the result is proved.

7. (a) Consider a polynomial with integer coefficients:

$$a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 . \qquad (a_n \neq 0) .$$

Prove that if $\frac{p}{q}$ is a rational root of this polynomial given in lowest terms, then p is a factor of a_0 and q is a factor of a_n.

If $\frac{p}{q}$ is a root of the polynomial, then

$$a_n \frac{p^n}{q^n} + a_{n-1} \frac{p^{n-1}}{q^{n-1}} + \dots + a_1 \frac{p}{q} + a_0 = 0 ,$$

whence, on multiplying by q^n,

$$a_n p^n + a_{n-1} p^{n-1} q + \dots + a_1 p q^{n-1} + a_0 q^n = 0 .$$

It follows that p is a factor of $a_0 q^n$ and q is a factor of $a_n p^n$. Since p and q have no factors in common, we conclude that p is a factor of a_0 and q a factor of a_n.

(b) Show that $x^3 + x + 1$ has no rational root.

By the preceding result the only conceivable rational roots are 1 and -1 and neither is a root.

(c) Prove that if $\sqrt{n}$ is rational then it is integral.

A rational root $\frac{p}{q}$ of $x^2 - n = 0$ must be an integer divisor of n , since $q = \pm 1$.

(d) Prove that $\sqrt{3} - \sqrt{2}$ is irrational.

Set $a = \sqrt{3} - \sqrt{2}$. Squaring we obtain

$$a^2 = 5 - 2\sqrt{6}$$

whence

$$(a^2 - 5) = -2\sqrt{6}$$

and

$$(a^2 - 5)^2 = 24$$

or

$$a^4 - 10a^2 + 1 = 0 .$$

The only conceivable rational roots of this equation are ± 1 and neither is a root.

Alternatively. Assume $\sqrt{3} - \sqrt{2} = r$ (rational); then,

$$r^2 - 2r\sqrt{2} + 2 = 3 .$$

Since this implies that $\sqrt{2}$ is rational, $\sqrt{3} - \sqrt{2}$ is irrational.

8. Prove that an ordered field in which the Nested Interval Principle holds also obeys the Separation Axiom.

Let A and B be two sets such that for each $\alpha \in A$ and $\beta \in B$, $\alpha \leq \beta$. If there exists $\alpha_s \in A$, $\beta_s \in B$, such that $\alpha_s = \beta_s = s$, then s separates A and B , from $\alpha \leq \beta_s \leq s$ and $\beta \geq \alpha_s \geq s$.

Now suppose all $\beta - \alpha > h$, $h > 0$. Take any β in B . Then some $s = \beta - nh'$, n a positive integer and $0 < h' < h$, will separate A and B .

There is one more case; for every ϵ, there exist $\alpha_\epsilon \varepsilon A$, $\beta_\epsilon \varepsilon B$. such that $\beta_\epsilon - \alpha_\epsilon < \epsilon$, but for no α, β does $\alpha = \beta$. We first find pairs of numbers (α_n,β_n) such that $\beta_n - \alpha_n < \frac{1}{n}$ and $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \cdots$, $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \geq \cdots$. Setting $\epsilon = 1$, there exist (α_1,β_1) such that $\beta_1 - \alpha_1 < 1$. Setting $\epsilon = \frac{1}{2}$, there exist (α_2,β_2) such that $\beta_2 - \alpha_2 < \frac{1}{2}$. Then, either $\alpha_2 > \alpha_1$ or $\beta_2 < \beta_1$. If $\alpha_2 > \alpha_1$ and $\beta_2 < \beta_1$, then we have found (α_1,β_1), (α_2,β_2) as desired. If either $\alpha_2 < \alpha_1$ or $\beta_2 > \beta_1$, say $\alpha_2 < \alpha_1$, then $\beta_2 - \alpha_1 < \frac{1}{2}$ and the pairs (α_1,β_1), (α_1,β_2) satisfy the condition. At every step, such a choice can be made, and so the pairs (α_n,β_n) exist.

Now by the Nested Interval Principle, there is some s contained in all $[\alpha_n,\beta_n]$. We claim s separates A and B. Else, either $s >$ all α and $s >$ some β or $s <$ all β and $s <$ some α, say the former; $s > \beta_s$. $\beta_s - \alpha \geq t > 0$ for all α. But for n large enough, $\beta_n - \alpha_n < \frac{1}{n} < t$. So $\beta_n < \beta_s < s$ and s is not in $[\alpha_n,\beta_n]$. Similarly for $s < \alpha_s$. So s is a separation number for A and B.

9. Prove that an ordered field in which the Least Upper Bound Principle holds also obeys the Separation Axiom.

Take two sets, A and B, for which every member of A is less than or equal to each member of B. By the Least Upper Bound Principle, there exists a least upper bound s for A such that $s \geq a$, $a \in A$ and $s \leq b$, b any other upper bound for A. In particular, $s \leq b$, $b \in B$, since each member of B is an upper bound for A. So s is a separation number for A and B and the Separation Axiom holds.

Teacher's Commentary

Appendix 2

FUNCTIONS AND THEIR REPRESENTATIONS

TC A2-1. Functions.

Functions whose domains and ranges are subsets of real numbers are usually called "real valued functions of a real variable." In Chapter 11 we shall talk about vector valued functions of a real variable. In Chapter 14 we shall discuss sequences of real numbers, i.e., real valued functions of an integer variable. In the same chapter we also have sequences of functions, i.e., function valued functions of an integer variable.

Example A2-1b. The statement that the range of the function $f : x \rightarrow x^2$ is the set of nonnegative real numbers is equivalent to the statement that every positive number p has a square root. This can be proved in the following way. Let A be the set of nonnegative numbers whose squares are less than p. A is not empty since 0 is in A. A is bounded above, say by $p + 1$. Thus, A has a least upper bound, s (Section A1-5). We cannot have $s^2 < p$ for, if $h < 1$, then

$$\begin{aligned}(s + h)^2 &= s^2 + 2sh + h^2 \\ &= s^2 + h(2s + h) \\ &< s^2 + h(2s + 1) \\ &< s^2 + h(2p + 3) .\end{aligned}$$

Thus, if

$$h < \min\{1 , \frac{p - s^2}{2p + 3}\} ,$$

then $(s + h)^2 < p$ and s could not be an upper bound for A. Similarly, $s^2 > p$ is impossible, for if

$$0 < h < \frac{s^2 - p}{2s} ,$$

then $(s - h)^2 > p$, and thus s could not be the least upper bound of A. Hence $s^2 = p$.

Example A2-1e. We can define many other functions whose graphs are contained in the graph of $x^2 + y^2 = 25$; we need only assign to each x either $\sqrt{25 - x^2}$ or $-\sqrt{25 - x^2}$. For example, for any a , $-5 \leq a < 5$, we have,

$$f_a : x \rightarrow \begin{cases} \sqrt{25 - x^2} , & -5 \leq x \leq a \\ -\sqrt{25 - x^2} , & a < x \leq 5 , \end{cases}$$

or

$$f_{unnny} : x \rightarrow \begin{cases} \sqrt{25 - x^2} , & x \text{ rational, } |x| \leq 5 \\ -\sqrt{25 - x^2} , & x \text{ irrational, } |x| < 5 . \end{cases}$$

However, the examples in the text are distinguished by the property that they are the only continuous functions defined on $[-5,5]$ whose graphs are contained in the graph of $x^2 + y^2 = 25$.

Solutions Exercises A2-1

1. Below are given examples of associations between elements of two sets. Decide whether each example may properly represent a function. This also requires you to specify the domain and range for each function. Note that no particular variable has to be the domain variable, and also some of the relations may give rise to several functions.

   Note that answers supplied are not necessarily the only correct ones. They are, like the examples, merely samples of the kind of ideas that are possible.

   (a) Assign to each nonnegative integer n the number $2n - 5$.

   This is a function with domain the set of nonnegative integers and range the set of odd integers not less than -5 .

   (b) Assign to each real number x the number 7.

   A constant function. Domain the set of real numbers. Range the set consisting of the one element, 7.

   (c) Assign to the number 10 the real number y .

   Not a function.

(d) **Assign to each pair of distinct points in the plane the distance between them.**

A function whose domain is the set of all pairs of distinct points in the plane and whose range is the set of positive real numbers.

(e) $y = -3$ (for all x)

May represent a function (constant), whose domain is the set of real numbers. The range is $\{-3\}$.

(f) $x = 4$ (for all y and z).

Not a function if x is considered the domain variable. It is a function if the set of ordered pairs (y,z) is considered the domain with y and z real numbers. The range is $\{4\}$.

(g) $x + y = 2$

A function, domain = $\{x : x$ is a real number$\}$ and
range = $\{y : y = 2 - x\}$, or vice versa.

(h) $y = 2x^2 + 3$

A function, with domain the set of all real numbers and range = $\{y : y \geq 3\}$.
If y is taken as the domain variable, the range set must be restricted to avoid ambiguity, and if the range is restricted to reals, the domain may have to be restricted to real numbers ≥ 3.

(i) $y^2 - 4 = x$

If x is an element of the domain, this equation does not define a function explicitly. But $f : x \rightarrow y = \sqrt{x + 4}$, $x \geq -4$, and $g : x \rightarrow y = -\sqrt{x + 4}$, $x \geq -4$, are functions whose ranges are the nonnegative and nonpositive real numbers, respectively. Also $h : y \rightarrow x = y^2 - 4$, $y \leq -2$ or $y \geq 2$, is a function whose range is $\{x : x \geq -4\}$.

(j) $y < 2x - 1$

Not a function.

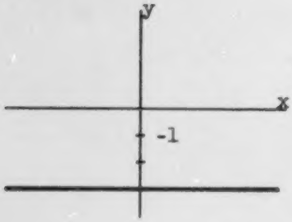
(k) $f(x) = -\sqrt{16 - x^2}$

A function. Usually, the domain is restricted to the set $\{x : -4 \leq x \leq 4\}$ so that the range will be real numbers, here the interval $[-4,0]$. Complex numbers are not used in this course in the calculus.

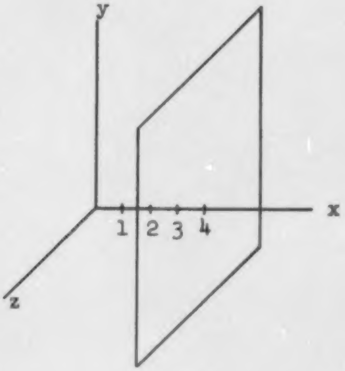
(ℓ) $x^2 + y^2 = 16$

This equation does not represent a function explicitly. See part (k) for one possible function obtained from this equation.

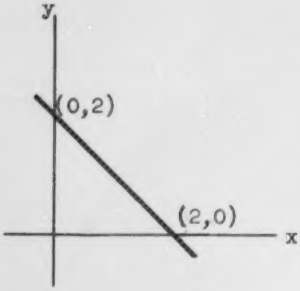
2. (e)



(f)



(g)



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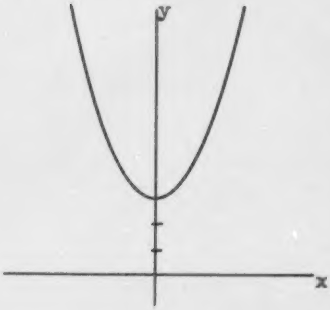
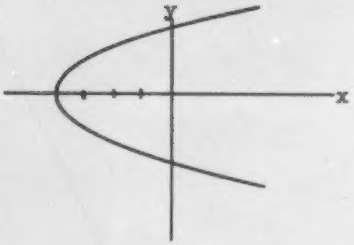


(i)



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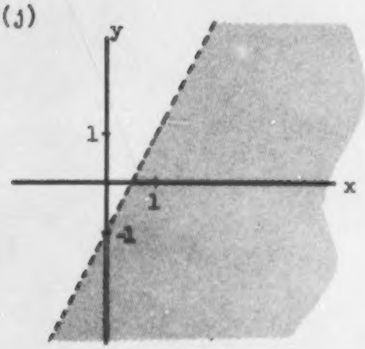

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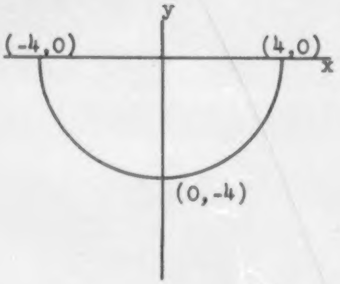
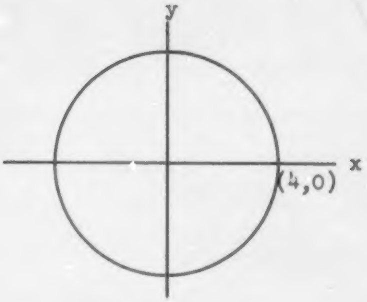


(ℓ)



3. A function f is completely defined by the table:

x	0	1	2	3	4
$f(x)$	-3	1	5	9	13

(a) Describe the domain and range of f .

Domain $= \{0 , 1 , 2 , 3 , 4\}$. Range $= \{-3 , 1 , 5 , 9 , 13\}$.

(b) Write an equation with suitably restricted domain that defines f .

$f(x) = 4x - 3$, x an integer, and $0 \leq x \leq 4$.

4. If $f : x \rightarrow x^2 + 3x - 4$, find

(a) $f(0) = -4$

(b) $f(2) = 6$

(c) $f(-1) = -6$

(d) $f(\sqrt{3}) = 3\sqrt{3} - 1$

(e) $f(2 - \sqrt{2}) = 8 - 7\sqrt{2}$

(f) $ff(1) = f(0) = -4$

5. If g is a function defined by $g(x) = \dfrac{2x}{\sqrt{5 - x^2}}$, find if possible:

(a) $g(0) = 0$

(b) $g(1) = 1$

(c) $g(-1) = -1$

(d) $g(2) = 4$

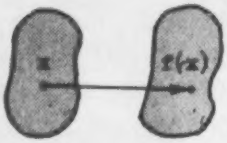
(e) $g(-3)$

(f) $g(\sqrt{5})$

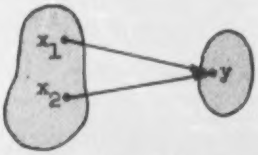
$\left.\begin{matrix} g(-3) \\ g(\sqrt{5}) \end{matrix}\right\}$ undefined.

6. Which of the following mappings represent functions?

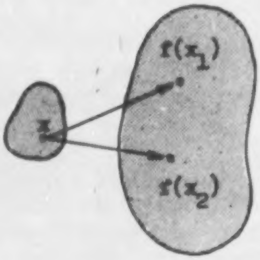
(a)



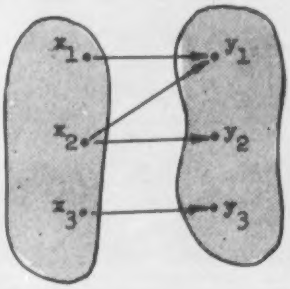
(d)



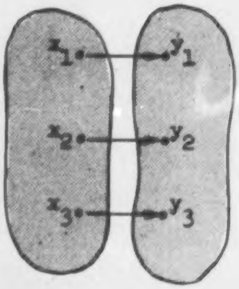
(b)



(e)



(c)



(a), (c) and (d) are functions.

7. Given the function $f : x \rightarrow x$ and $g : x \rightarrow \dfrac{x^2}{x}$. If x is a real number, are f and g the same function? Why or why not?

They are NOT the same function, since $x = 0$ is not in the domain of g but is in f.

8. Given the functions $f : x \rightarrow x + 2$ and $g : x \rightarrow \frac{x^2 - 4}{x - 2}$. If x is real, are f and g the same function? Why or why not?

They are not the same function, since $x = 2$ is not in the domain of g but is in f.

9. What number or numbers have the image 10 under the following mappings?

(a) $f : x \rightarrow 2x$	$x = 5$
(b) $g : x \rightarrow x^2$	$x = \pm \sqrt{10}$
(c) $h : x \rightarrow \sqrt{x^2 + 36}$	$x = \pm 8$
(d) $\alpha : x \rightarrow \lvert x - 4 \rvert$	$x = 14, -6$
(e) $\phi : x \rightarrow [x]$	$10 \leq x < 11$

10. Which of the following statements are always true for any function f, assuming that x_1 and x_2 are in the domain of f?

(a) If $x_1 = x_2$, then $f(x_1) = f(x_2)$.
(b) If $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.
(c) If $f(x_1) = f(x_2)$, then $x_1 = x_2$.
(d) If $f(x_1) \neq f(x_2)$, then $x_1 \neq x_2$.

(a) and (d) are always true.

11. If $f(x) = |x|$, which of the following statements are true for all real numbers x and t?

(a) f is an odd function.
(b) $f(x^2) = \left(f(x)\right)^2$
(c) $f(x - t) \leq f(x) - f(t)$
(d) $f(x + t) \leq f(x) + f(t)$

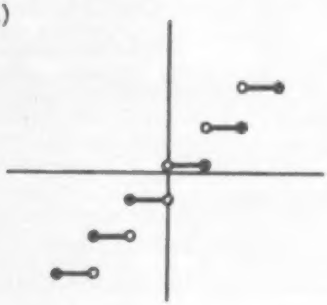
(b) and (d) are true statements.

12. Which of the following functions are even, which are odd, and which are neither even nor odd?

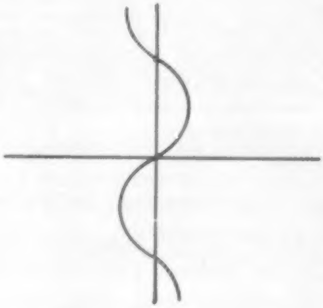
(a)	$f : x \rightarrow 3x$	odd
(b)	$f : x \rightarrow -2x^2 + 5$	even
(c)	$f : x \rightarrow x^2 - 4x + 4$	neither
(d)	$f : x \rightarrow -2x + 1$	neither
(e)	$f : x \rightarrow x^3 + 4$	neither
(f)	$f : x \rightarrow x^3 - 2x$	odd
(g)	$f : x \rightarrow 2^{1/x}$	neither
(h)	$f : x \rightarrow 2^{1/x^2}$	even

13. Which of the following graphs could represent functions?

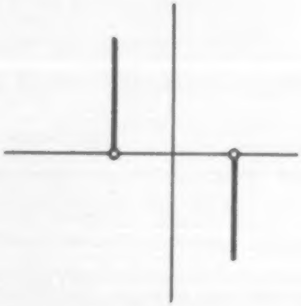
(a)



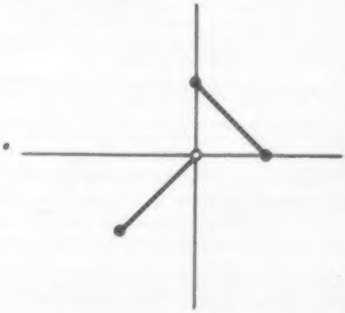
(c)



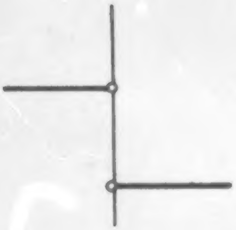
(b)



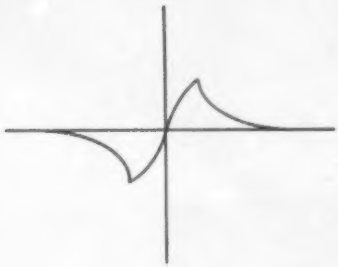
(d)



(e)

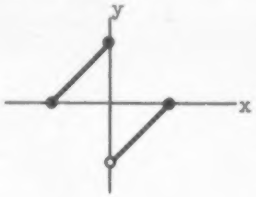


(f)



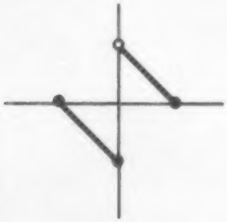
a , d , e and f could represent functions.

14. Suppose that $f : x \rightarrow f(x)$ is the function whose graph is shown.

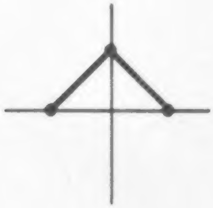


Sketch the graphs of

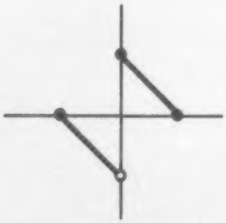
(a) $g : x \rightarrow -f(x)$



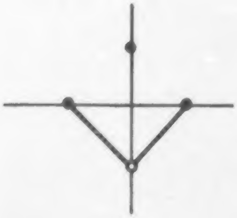
(c) $g : x \rightarrow |f(x)|$



(b) $g : x \rightarrow f(-x)$



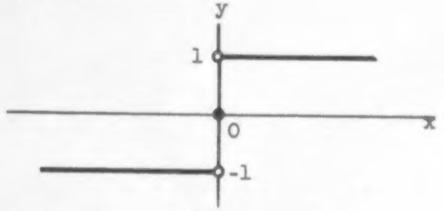
(d) $g : x \rightarrow f(|x|)$



15. A function f is defined by $f(x) = \begin{cases} \frac{x}{|x|} & \text{for } x \neq 0 , \\ 0 & \text{for } x = 0 . \end{cases}$

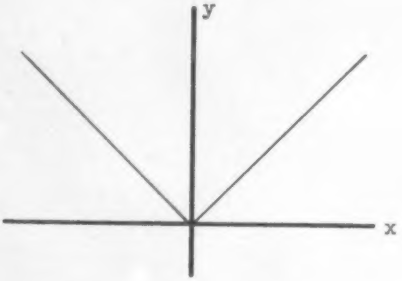
Identify this function and sketch its graph.

The signum function.



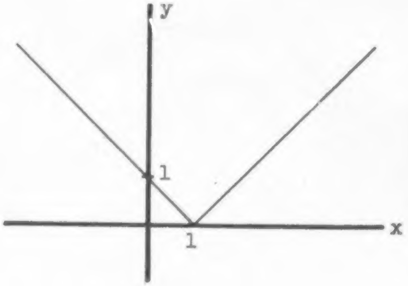
16. Sketch the graph of each function, specifying its domain and range. In each case, the domain is the set of all real numbers.

(a) $f : x \rightarrow \sqrt{x^2}$



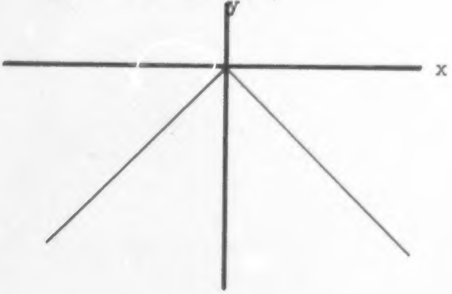
Range: all nonnegative reals.

(c) $f(x) = |1 - x|$



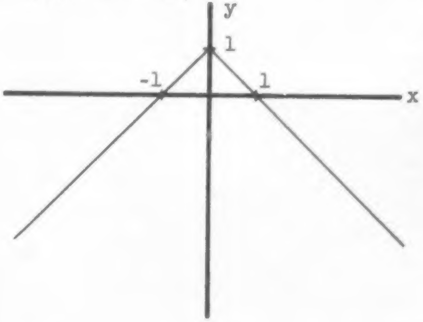
Range: all nonnegative reals.

(b) $f : x \rightarrow -|x|$



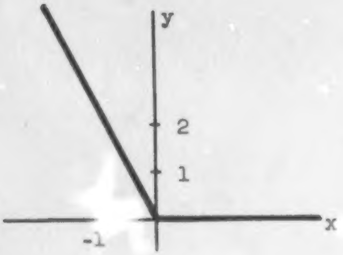
Range: all nonpositive reals.

(d) $f(x) = 1 - |x|$



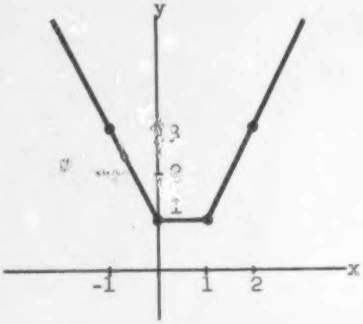
Range = $\{y : y \leq 1\}$

(e) $f : x \rightarrow |x| - x$



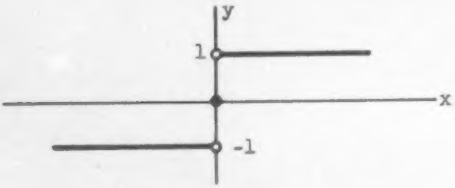
Range: all nonnegative reals.

(f) $f : x \rightarrow |x| + |x - 1|$
(Hint: Consider separately the three possibilities: $x < 0$, $0 \leq x \leq 1$, and $x > 1$.)



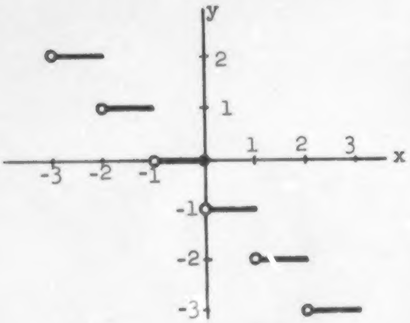
Range = $\{y : y \geq 1\}$

(g) $f : x \rightarrow \text{sgn}\, x$



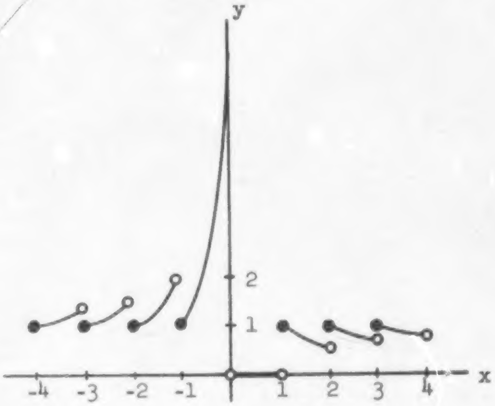
Range = $\{-1 , 0 , 1\}$

(h) $f : x \rightarrow [-x]$



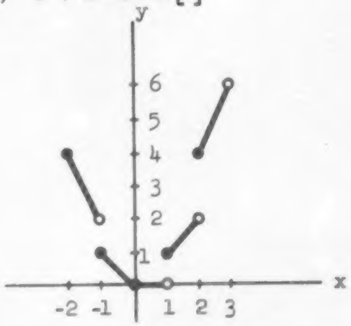
Range: all integers.

(i) $f : x \rightarrow \frac{[x]}{x}$



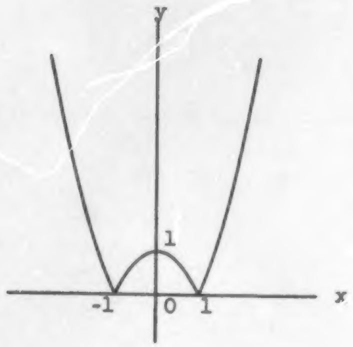
Range = $\{y : y = 0$ or $y > \frac{1}{2}\}$

(j) $f : x \rightarrow x[x]$



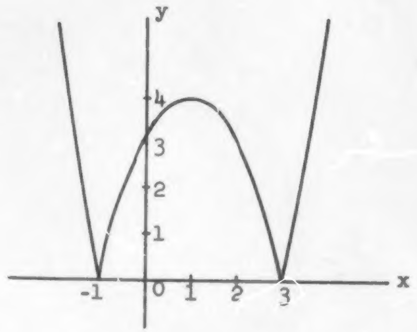
Range = $\{y : y \geq 0$ except $y = n^2 - n$, where n is an integer $\geq 2\}$

(k) $f : x \rightarrow |1 - x^2|$



Range = $\{y : y \geq 0\}$

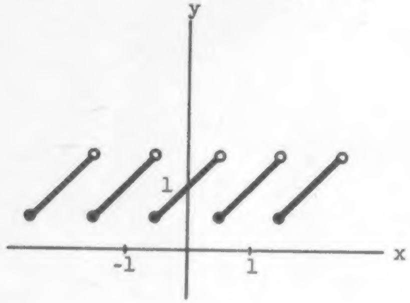
(ℓ) $f : x \rightarrow |x^2 - 2x - 3|$



Range = $\{y : y \geq 0\}$

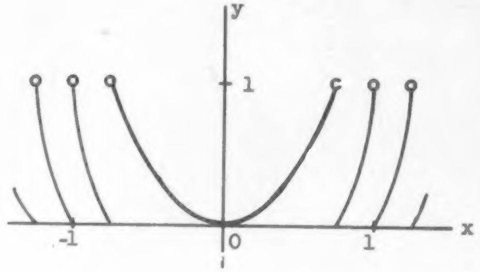
Sketch the graphs of the functions in Exercises 17 to 19. For those functions which are periodic, indicate their period. Indicate those functions which are even or odd.

17. (a) $f : x \rightarrow x - \left[x - \frac{1}{2}\right]$



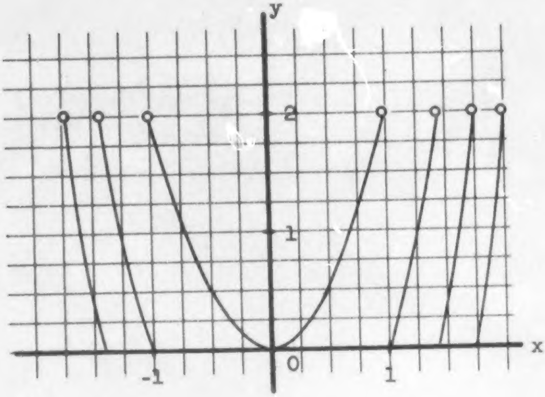
Periodic. $p = 1$

(b) $f : x \rightarrow 2x^2 - \left[2x^2\right]$

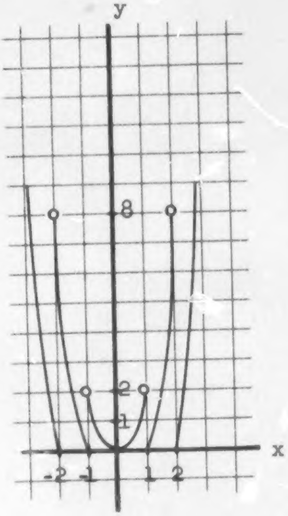


Function is EVEN.

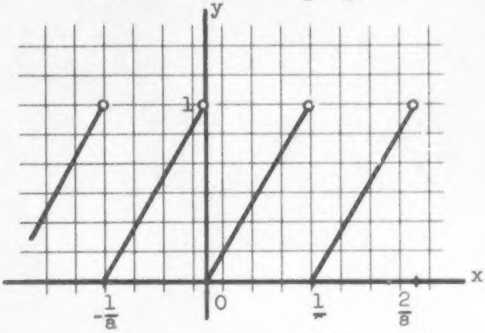
(c) $f : x \longrightarrow 2x^2 - 2[x^2]$



(d) $f : x \longrightarrow 2x^2 - 2[x]^2$



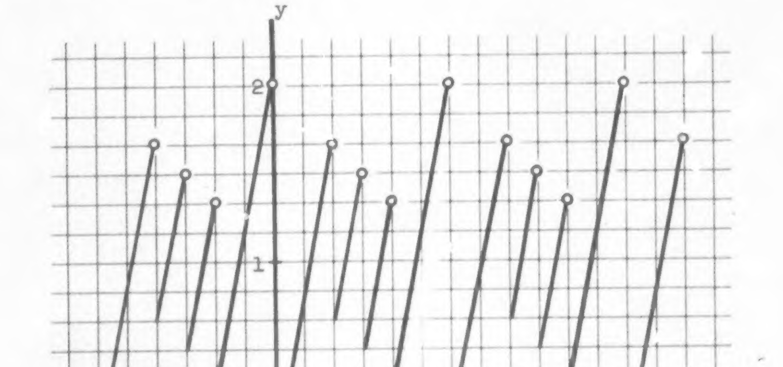
18. (a) $f : x \longrightarrow a - [ax]$, $a > 0$



Periodic: $p = \frac{1}{a}$

Note that the slope of each piece is a .

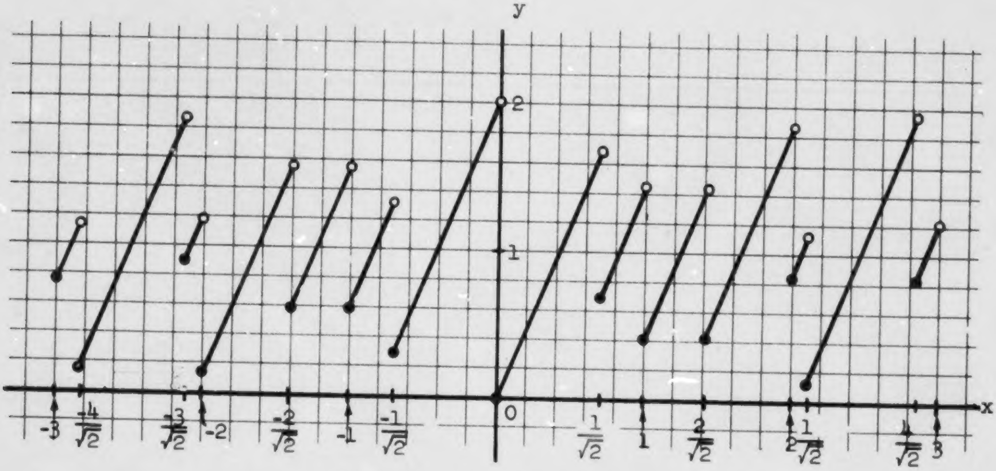
(b) $f : x \longrightarrow 5x - [2x] - [3x]$



Note that $f(x) = g(x) + h(x)$ where $g(x) = 2x - [2x]$ and $h(x) = 3x - [3x]$, two periodic functions like that of part (a). Functions g and h have periods $\frac{1}{2}$ and $\frac{1}{3}$, respectively; hence $g + h$ is periodic and its period is the least common integer multiple of $\frac{1}{2}$ and $\frac{1}{3}$, which is 1; i.e., $2 \cdot \frac{1}{2} = 3 \cdot \frac{1}{3} = 1$. In general, if two periodic functions f_1 and f_2 have periods p_1 and p_2, respectively (p_1, p_2, positive real numbers), and if there exist integers n_1 and n_2 such that $n_1 p_1 = n_2 p_2$, then $f_1 + f_2$ is periodic with period $n_1 p_1$.

Note also that the slope of each piece of the graph of f is 5.

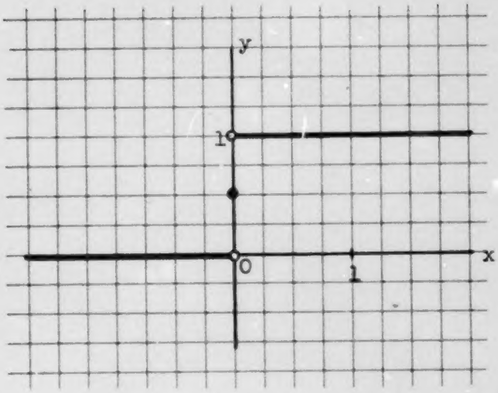
(c) $f : x \rightarrow x(\sqrt{2} + 1) - [x\sqrt{2}] - \lfloor x \rfloor$



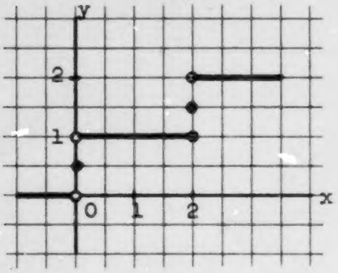
In this case, $f(x) = g(x) + h(x)$ where $g(x) = x\sqrt{2} - [x\sqrt{2}]$ and $h(x) = x - [x]$. The periods of g and h are $\frac{3}{\sqrt{2}}$ and 1 respectively, and since these numbers are incommensurable, $f = g + h$ is not a periodic function. Note that the points of discontinuity occur at the integers and at integer multiples of $\frac{1}{\sqrt{2}}$. The slope of each piece of the graph of f is $1 + \sqrt{2}$.

19. (a) $f : x \rightarrow \dfrac{1 + \operatorname{sgn} x}{2}$

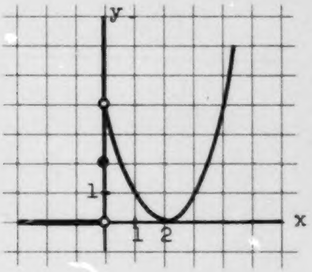
This function is also called the Heaviside Unit function and is designated by $f : x \rightarrow H(x)$.



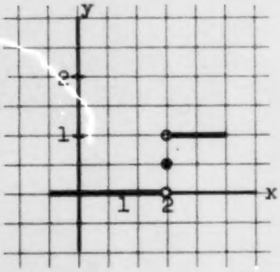
(b) $f : x \rightarrow H(x) + H(x - 2)$



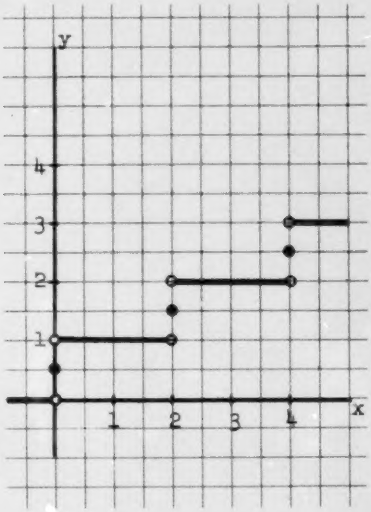
(d) $f : x \rightarrow (x - 2)^2 \cdot H(x)$



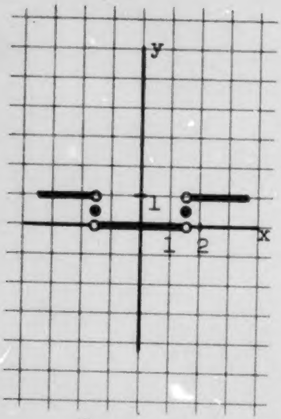
(c) $f : x \rightarrow H(x) \cdot H(x - 2)$



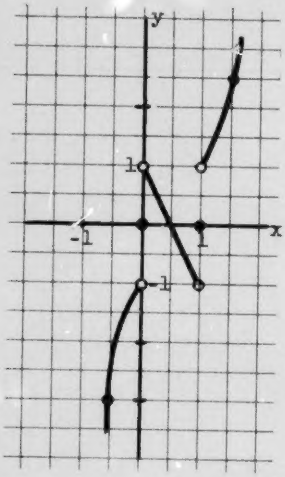
(e) $f : x \rightarrow H(x) + H(x - 2) + H(x - 4)$



(f) $f : x \longrightarrow H(x^2 - 2)$



(g) $f : x \longrightarrow (\operatorname{sgn} x)(x - 1)^2 + [\operatorname{sgn}(x - 1)]\, x^2$



20. If f and g are periodic functions of periods m and n, respectively, (m and n, integers), show that $f + g$, and $f \cdot g$ are also periodic. Give examples to show that the period of $f + g$ can either be greater or less than both of m and n. Repeat the same for the product $f \cdot g$. (Note: The problem in the text lacks the condition "m and n, integers.")

Since any integer multiple of a period is also a period, $f + g$ and $f \cdot g$ are periodic with period mn. However, this may or may not be the fundamental period.

If we choose $f(x) = \sin \pi x$ (period 2) and $g(x) = \sin \frac{2\pi x}{3}$ (period 3), then $f + g$ and $f \cdot g$ both have period 6.

If, however, we choose $f(x) = \sin \frac{2\pi x}{3}$ (period 3) and $g(x) = \sin \pi x - \sin \frac{2\pi x}{3}$ (period 6), then $f + g$ has period 2.

If $f(x) = 2 \sin \pi x$ (period 2) and $g(x) = \cos \pi x$ (period 2), then $f(x) \cdot g(x) = \sin 2\pi x$ which has period 1.

21. (a) Can a function be both even and odd?

Yes; only $f : x \to 0$.

(b) What can you say about the evenness or oddness of the product of:

(1)	an even function by an even function?	EVEN
(2)	an even function by an odd function?	ODD
(3)	an odd function by an odd function?	EVEN

(c) Show that every function whose domain contains $-x$ whenever it contains x can be expressed as the sum of an even function plus an odd function.

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{ODD}}$$

22. Find functions $f(x)$ satisfying $f(x) \cdot f(-x) = 1$ (called a functional equation).
Suggestion: Use Number 21(c).

Let $f(x) = E(x) + \theta(x)$ where E is even and θ is odd. Then $f(-x) = E(x) - \theta(x)$. Using the given condition, we have $f(x) \cdot f(-x) = E^2(x) - \theta^2(x) - 1$. Hence, $E(x) = \sqrt{1 + \theta^2(x)}$ and $f(x) = \sqrt{1 + \theta^2(x)} + \theta(x)$, where θ is an arbitrary odd function. For example,

$$\theta(x) = x \quad , f(x) = \sqrt{1 + x^2} + x ;$$

$$\theta(x) = \tan x , f(x) = \sec x + \tan x .$$

Alternatively. Let $f(0) = 1$ or -1 and let $f(x)$ be an arbitrary nonzero function for $x > 0$. Then define $f(-x) = \frac{1}{f(x)}$ for $x > 0$. This construction gives the entire class of functions satisfying the given functional equation.

23. Prove that no periodic function other than a constant can be a rational function. (Note: A rational function is the ratio of two polynomial functions.)

Our proof is indirect. Assume that there exists a rational function $\frac{P(x)}{Q(x)}$ (P, Q, polynomials) which has a period $m > 0$. Let k denote the value of the function at $x = a$ where $Q(a) \neq 0$. Now consider the polynomial equation $P(x) - kQ(x) = 0$. This polynomial vanishes for $x = a + nm$, $n = 0, 1, 2, \ldots$ Since a polynomial of degree d can have at most d roots, we get a contradiction, and thus our initial assumption is false.

TC A2-2. Composite Functions.

In the introduction to this section it is suggested that the operation of composition for functions has no counterpart in the algebra of numbers. However, the set $P(A)$, consisting of all one-to-one mappings of the set A onto itself, with the operation of composition has the following properties (we denote the inverse of f by f^{-1} and the identity function by I):

(i) if $g, h, \in P(A)$, then $gh \in P(A)$.

(ii) $(fg)h = f(gh)$ for all $f, g, h, \in P(A)$.

(iii) $fI = If = f$ for all $f \in P(A)$.

(iv) $ff^{-1} = f^{-1}f = I$ for all $f \in P(A)$.

These are the postulates for a group. Thus, $P(A)$ shares the same algebraic structure with the positive reals under multiplication, with the reals under addition, and many other familiar groups.

Solutions Exercises A2-2

1. Given that $f : x \rightarrow x - 2$ and $g : x \rightarrow x^2 + 1$ for all real x, find

   (a) $f(2) + g(2) = 5$

   (b) $f(2) \cdot g(2) = 0$

   (c) $fg(2) = 3$

   (d) $gf(2) = 1$

   (e) $f(x) + g(x) = x^2 + x - 1$

   (f) $f(x) \cdot g(x) = x^3 - 2x^2 + x - 2$

   (g) $fg(x) = x^2 - 1$

   (h) $gf(x) = x^2 - 4x + 5$

2. If $f(x) = 3x + 2$ and $g(x) = 5$, find

   (a) $fg(x) = f(5) = 17$

   (b) $gf(x) = g(3x + 2) = 5$

3. If $f(x) = 2x + 1$ and $g(x) = x^2$, find $fg(x)$ and $gf(x)$. For what values of x, if any, are $fg(x)$ and $gf(x)$ equal?

   $fg(x) = 2x^2 + 1$; $gf(x) = 4x^2 + 4x + 1$.

   $fg = gf$ when $2x^2 + 1 = 4x^2 + 4x + 1$, or $2x^2 + 4x = 0$, or

   $x = 0$, or -2 .

4. For each pair of functions f and g, find the composite functions fg and gf and specify the domain (and range, if possible) of each.

   (a) $f : x \rightarrow \frac{1}{x}$; $g : x \rightarrow 2x - 6$

   (b) $f : x \rightarrow \frac{1}{x}$; $g : x \rightarrow x^2 - 4$

   (c) $f : x \rightarrow \frac{1}{x}$; $g : x \rightarrow \sqrt{x}$

   (d) $f : x \rightarrow x^2$; $g : x \rightarrow \sqrt{x}$

   (e) $f : x \rightarrow x^2$; $g : x \rightarrow \sqrt{4 - x}$

   (f) $f : x \rightarrow -x^2$; $g : x \rightarrow \sqrt{x}$

	fg	Domain	Range	gf	Domain	Range
(a)	$\frac{1}{2x - 6}$	$x \neq 3$	$y \neq 0$	$\frac{2}{x} - 6$	$x \neq 0$	$y \neq -6$
(b)	$\frac{1}{x^2 - 4}$	$x \neq 2, -2$	$y \neq 0$	$\frac{1}{x^2} - 4$	$x \neq 0$	$y \neq -4$
(c)	$\frac{1}{\sqrt{x}}$	$x > 0$	$y > 0$	$\sqrt{\frac{1}{x}}$	$x > 0$	$y > 0$
(d)	x	$x \geq 0$	$y \geq 0$	$\lvert x \rvert$	all x	$y \geq 0$
(e)	$4 - x$	$x \leq 4$	$y \geq 0$	$\sqrt{4 - x^2}$	$[-2,2]$	$[0,2]$
(f)	$-x - 1$	$x \geq 0$	$y \leq -1$	Not a real function		

5. Given that $f(x) = x^2 + 3$ and $g(x) = \sqrt{x + 2}$, solve the equation

$$fg(x) = gf(x) .$$

$fg(x) = |x + 2| + 3 = \sqrt{x^2 + 5} = gf(x)$ if and only if $x^2 + 4x + 4 + 6|x + 2| + 9 = x^2 + 5$ or $4x + 8 + 6|x + 2| = 0$, whence $x = -2$.

Note: The use of "if and only if" requires absolute value.

Alternatively. Solve $x + 5 = \sqrt{x^2 + 5}$ for the set of possible solutions and then check the results in the original equation.

6. Solve Problem 5 taking $g(x) = \sqrt{x - 2}$.

No solutions.

7. Describe functions f and g such that gf will equal:

(a) $3(x + 2) - 4$ $\quad f(x) = x + 2 \quad ; \ g(x) = 3x - 4$

(b) $(2x - 5)^3$ $\quad f(x) = 2x - 5 \quad ; \ g(x) = x^3$

(c) $\frac{3}{2x - 5}$ $\quad f(x) = 2x - 5 \quad ; \ g(x) = \frac{3}{x}$

(d) $\sqrt{x^2 - 4}$ $\quad f(x) = x^2 - 4 \quad ; \ g(x) = \sqrt{x}$

(e) $(x^4)^2$ $\quad f(x) = x^4 \quad ; \ g(x) = x^2$

The answer is not unique. A simple answer may be $f(x) = x$ and $g(x)$ equal to functions shown. The functions shown above are typical of those used later.

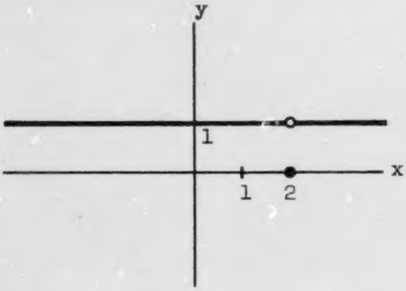
8. For each pair of functions f and g find the composite functions fg and gf and specify the domain (and range, if possible) of each. Also, sketch the graph of each, and give the period (fundamental) of those which are periodic.
(a) $f : x \longrightarrow |x|$; $g : x \longrightarrow \operatorname{sgn}(x - 2)$

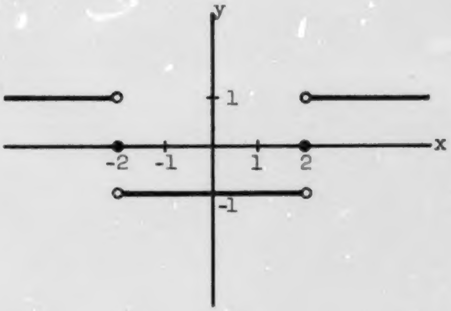
$fg(x) = |\operatorname{sgn}(x - 2)|$

Domain: all reals;
Range: $\{0,1\}$

$gf(x) = \operatorname{sgn}(|x| - 2)$

Domain: all reals;
Range: $\{-1,0,1\}$





(b) $f : x \rightarrow |x|$; $g : x \rightarrow 2\operatorname{sgn}(x - 2) - 1$

$fg(x) = |2 \operatorname{sgn}(x - 2) - 1|$

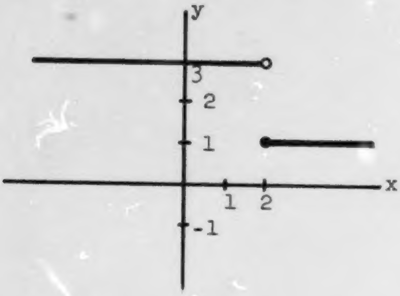
Domain: all reals;

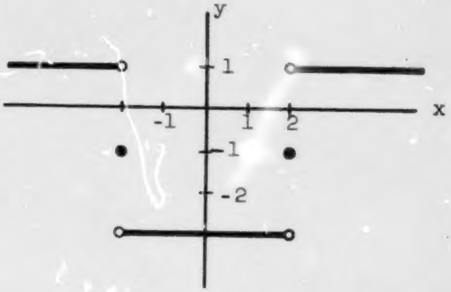
Range: $\{1,3\}$

$gf(x) = 2\operatorname{sgn}(|x| - 2) - 1$

Domain: all reals;

Range: $\{1,-1,-3\}$





9. What cay you say about the evenness or oddness of the composite of an

(a) even function of an even function? EVEN

(b) even function of an odd function? EVEN

(c) odd function of an odd function? ODD

(d) odd function of an even function? EVEN

10. If the function f is periodic, what can you say about the periodic character of the composite functions fg and gf assuming these exist. g is an arbitrary function (not periodic). Illustrate by examples.

gf has the same periods as f and may have a smaller fundamental period.

fg need not be periodic.

Example:

$f : x \rightarrow \sin x$, period 2π ;

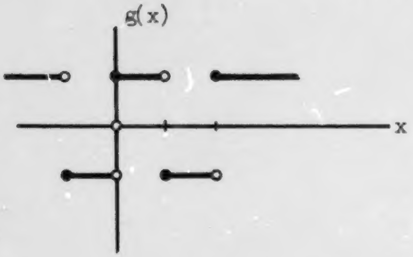
$g : x \rightarrow x^2$.

$gf : x \rightarrow (\sin x)^2$, period π ;

$fg : x \rightarrow \sin x^2$.

11. If the functions f and g are each periodic, then the composite functions fg and gf (assumed to exist) are also periodic. Can the period of either one be less than that of both f and g ?

Yes. If $f(x) = \sin \pi x$, and if g is defined by



$$g(x) = \begin{cases} 1 , & 0 \le x < 1 \\ -1 , & 1 \le x < 2 \end{cases}$$

and $g(x + 2) = g(x)$, (g is called the square wave function). Then $fg(x) = 0$ for all x , and fg has any period $p > 0$. (The period of gf is 1 .)

Λ12. A sequence $a_0 , a_1 , a_2 , \dots , a_n , \dots$ is defined by the equation

$$a_{n+1} = f(a_n) , \quad n = 0 , 1 , 2 , 3 , \dots$$

where f is a given function and a_0 is a given number. If $a_0 = 0$ and $f : x \rightarrow \sqrt{2 + x}$, then

$$a_1 = f(a_0) = \sqrt{2} ,$$

$$a_2 = f(a_1) = ff(a_0) = \sqrt{2 + \sqrt{2}} ,$$

$$a_3 = f(a_2) = ff(a_1) = fff(a_0) = \sqrt{2 + \sqrt{2 + \sqrt{2}}} .$$

Show that for any n ,

(a) $a_n < 2$.

By Induction:

For $n = 1$, $a_1 = \sqrt{2 + a_0} = \sqrt{2} < 2$.

Assume $a_n < 2$ for $n = k$.

Then
$$a_{k+1} = \sqrt{2 + a_k} < \sqrt{2 + 2}$$
$$< \sqrt{4}$$
$$< 2$$

Q.E.D.

(b) $a_n > 2 - \frac{1}{2^{n-1}}$, $n > 0$.

The statement is true for $n = 1$. Suppose it is true for $n = k$. Then

$$a_{n+k} = \sqrt{2 + a_k}$$

$$(a_{k+1})^2 - 4 = a_k - 2$$

$$2 - a_{k+1} = \frac{2 - a_k}{2 + a_{k+1}} < \frac{2 - a_k}{2}$$

$$< \frac{2 - (2 - \frac{1}{2^{k-1}})}{2}$$

$$\leq \frac{1}{2^k} ,$$

from which the conclusion follows.

A13. If $a_{n+1} = f(a_n)$, $n = 0 , 1 , 2 , \ldots$, $a_0 = \mu$, find a_n as a function of μ and n , for the following functions f :

(a) $f : x \rightarrow a + bx$

$$a_0 = \mu$$

$$a_1 = f(a_0) = a + b\mu$$

$$a_2 = f(a_1) = a + b(a + b\mu)$$

$$\cdots$$

$$a_n = f(a_{n-1}) = a + ab + ab^2 + \ldots + ab^{n-1} + b^n\mu$$

(b) $f : x \rightarrow x^m$

$$a_0 = \mu$$

$$a_1 = \mu^m$$

$$a_2 = (\mu^m)^m = \mu^{(m^2)}$$

$$\cdots$$

$$a_n = (\mu^{m^{n-1}})^m = \mu^{(m^n)}$$

(c) $f : x \rightarrow \sqrt{|x|}$

$$a_0 = \mu$$

$$a_1 = \sqrt{|\mu|} = |\mu|^{1/2} > 0$$

$$a_2 = (|\mu|^{1/2})^{1/2} = |\mu|^{1/4}$$

$$\cdots$$

$$a_n = |\mu|^{1/2^n}$$

(d) $f : x \rightarrow \sqrt{1 - x^2}$

$$a_0 = \mu$$

$$a_1 = \sqrt{1 - \mu^2}$$

$$a_2 = \sqrt{1 - (1 - \mu^2)} = |\mu|$$

$$\cdots$$

$a_{2n} = |\mu|$ and $a_{2n+1} = \sqrt{1 - \mu^2}$ for $n > 0$.

(e) $f : x \rightarrow (1 - x)^{-1}$

$$a_0 = \mu$$

$$a_1 = \frac{1}{1 - \mu}$$

$$a_2 = \frac{1}{1 - \frac{1}{1 - \mu}} = \frac{1 - \mu}{-\mu} = 1 - \frac{1}{\mu}$$

$$a_3 = \frac{1}{1 - (1 - \frac{1}{\mu})} = \frac{1}{\frac{1}{\mu}} = \mu$$

$$\cdots$$

$a_{3n} = \mu$, $a_{3n+1} = \frac{1}{1 - \mu}$, $a_{3n+2} = 1 - \frac{1}{\mu}$.

TC A2-3. Inverse Functions.

Given any set A, a relation R on A is a rule which permits us to say for any pair x,y in A whether or not x has or has not the relation R to y. If x has the relation R to y then we write $x\,R\,y$; if not, we write $x \not R\, y$. Thus for the relation $<$ we have $1 < 2$ while $2 \not< 1$. The graph of a relation is the set of ordered pairs (x,y) for which $x\,R\,y$. Thus, the graph of the relation $<$ is the set of points above the line $y = x$. Consider the relation $x\,R\,y$ if and only if $x^2 + y^2 = 25$. The graph of this relation is the circle of radius 5 with center at the origin.

> **DEFINITION A2-3b.** We have defined the inverse of a function only when the function is one-to-one. Moreover, we have required that the domain of the inverse of f be the range of f. Thus, although $\sin(\arcsin x) = x$ for all x in the domain of arcsine, sine is not the inverse of arcsine (i.e., $\arcsin(\sin x) = x$ is not true for all x). For these reasons it is essential that the definition of a function carry with it the domain of the function. Thus the functions
>
> $$f : x \rightarrow \sin x \qquad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$
>
> $$g : x \rightarrow \sin x \qquad x \in R$$
>
> are distinct, even though we may (by a convenient abuse of language) call them both by the same name, sine.

The situation that we have just described applies generally. Given any function f there is a function g, defined on the range of f, satisfying $fg(x) = x$ for each x in the range of f. We simply require $g : x \rightarrow$ any preimage of x (one must exist since x is in the range of f). Other authors sometimes designate g as a right inverse of f and f as a left inverse of g. The one-to-oneness of f is equivalent to the requirement that f has both a right and left inverse or the requirement that f has a unique right inverse.

Solutions Exercises A2-3

1. What is the reflection of the line $y = f(x) = 3x$ in the line $y = x$? Write an equation defining the inverse of f .

$$x = 3y \qquad \text{or} \qquad y = \frac{x}{3} .$$

2. Which points are their own reflections in the line $y = x$? What is the graph of all such points?

Those points with ordinate equal to abscissa. The graph is the line $y = x$.

3. (a) Find the slope of the segment from (a,b) to (b,a) , and prove that the segment is perpendicular to the line $y = x$.

Slope of line from (a,b) to (b,a) is

$$\frac{a - b}{b - a} = -1 .$$

Slope of $y = x$ is 1 .

Since product of slopes $= -1$, the lines are perpendicular.

(b) Prove that the segment from (a,b) to (b,a) is bisected by the line $y = x$.

Let the point of intersection be (x,x) ; then

$$\sqrt{(a - x)^2 + (b - x)^2} = \sqrt{(x - b)^2 + (x - a)^2} .$$

4. What is the reflection of $(1,1)$ in the line

(a) $x = 0$?	$(-1,1)$
(b) $y = 0$?	$(1,-1)$
(c) $y = -x$?	$(-1,-1)$
(d) $y = 2$?	$(1,3)$
(e) $x = -3$?	$(-7,1)$

5. Describe any function or functions you can think of which are their own inverses.

Any function whose graph has $y = x$ as an axis of symmetry; e.g.,

$$f(x) = x\ ,\ c - x\ ,\ \frac{1}{x}\ ,\ -\frac{1}{x}\ ,$$

$$\sqrt{1 - x^2} \text{ for } 0 \leq x \leq 1\ .$$

6. An equation or an expression (phrase) is said to be symmetric in x and y if the equations or the expressions remain unaltered by interchanging x and y ; e.g., $x^2 + y^2 = 0$, $x^3 + y^3 - 3xy$, $|x - y| = |x + y|$, $x - xy + y$. It follows that graphs of symmetric equations are symmetric about the $y = x$ line.
Geometrically, we can consider the line $y = x$ behaving as a mirror, i.e., for any portion of the graph there must also be a portion which is the mirror image. The equation

$$x^4 + y^4 = a^4$$

is obviously symmetric with respect to the line $y = x$. What other axes of symmetry (mirror type) does it have?

The x and y axes and the line $y = -x$. (Note: The graph is not a circle.)

7. The expression

$$a + b + |a - b| + 2c + \Big|a + b + |a - b| - 2c\Big|$$

is obviously symmetric in a and b . Show that it is also symmetric in a and c .

We have to show that

$$a + b + |a - b| + 2c + \Big|a + b + |a - b| - 2c\Big| = c + b + |c - b| + 2a + \Big|c + b + |c - b| - 2a\Big| .$$

Consider the six possible order relations

(1) $a \geq b \geq c$,

(2) $a \geq c \geq b$,

(3) $b \geq c \geq a$, etc.

For (1) and (2) we have $4a = 4a$; for (3), $4b = 4b$; and similarly for the other cases.

Another solution follows by noting that the given expression is also $4 \max\{a,b,c\}$, which is obviously symmetric in all three letters a , b , c , as well as any pair. (Compare with Exercises A1-3, No. 7b.)

8. Find the inverse of each function:

(a)	$f : x \to 3x + 6$	$g : x \to \frac{x - 6}{3}$
(b)	$f : x \to x^3 - 5$	$g : x \to (x + 5)^{1/3}$
(c)	$f : x \to \frac{2}{x} - 3$	$g : x \to \frac{2}{x + 3}$

9. Which of the following functions have inverses? Describe each inverse by means of a graph or equation and give its domain and range.

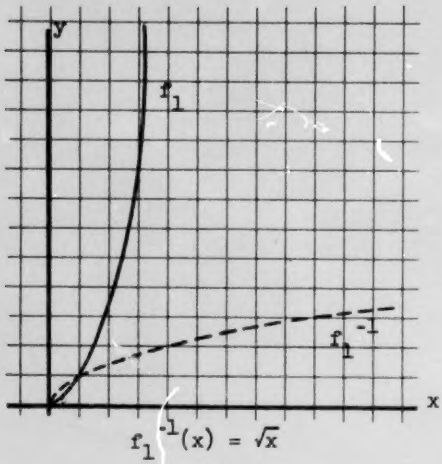
(a)	$f : x \to x^2$	Has no inverse.
(b)	$f : x \to \sqrt{x}$	$g : x \to x^2$ Domain and range: all nonnegative real numbers.
(c)	$f : x \to \lvert x \rvert$	Has no inverse.
(d)	$f : x \to [x]$	Has no inverse.
(e)	$f : x \to x\lvert x \rvert$	$g : x \to \sqrt{\lvert x \rvert}\ \mathrm{sgn}\ x$ Domain and range: all real numbers.
(f)	$f : x \to \mathrm{sgn}\ x$	Has no inverse.

10. As we have seen, $f : x \to x^2$ for all real x does not have an inverse. Do the following:

(a) Sketch graphs of $f_1 : x \to x^2$ for $x \geq 0$ and $f_2 : x \to x^2$ for $x \leq 0$, and determine the inverses of f_1 and f_2.

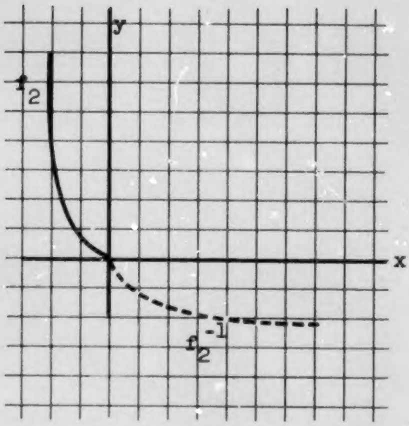
(b) What relationship exists among the domains of f, f_1, and f_2? (f_1 is called the restriction of f to the domain $\{x : x \geq 0\}$, and f_2 is similarly the restriction of f to the domain $\{x : x \leq 0\}$.)

(a)



$f_1^{-1}(x) = \sqrt{x}$

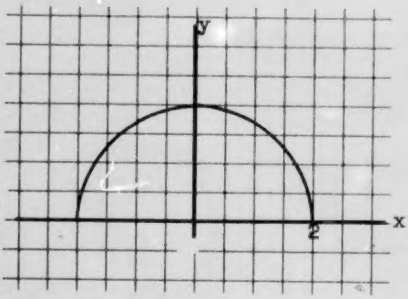
(b)



$f_2^{-1}(x) = -\sqrt{\ }$

Domain of f = Domain of $f_1 \cup$ Domain of f_2 .

11. (a) Sketch a graph of $f : x \to \sqrt{4 - x^2}$ and show that f does not have an inverse.



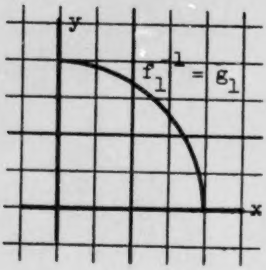
Each image (except 2) has two preimages; hence, f does not have an inverse.

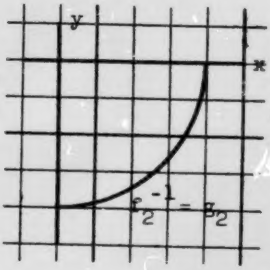
(b) Divide the domain of f into two parts such that the restriction of f to either part has an inverse.

$$\begin{cases} f_1 : x \to \sqrt{4 - x^2} & \text{Domain} = \{x : 0 \le x \le 2\} \\ f_2 : x \to \sqrt{4 - x^2} & \text{Domain} = \{x : -2 \le x \le 0\} \end{cases}$$

(c) Write an equation defining each inverse of part (b) and sketch the graphs.

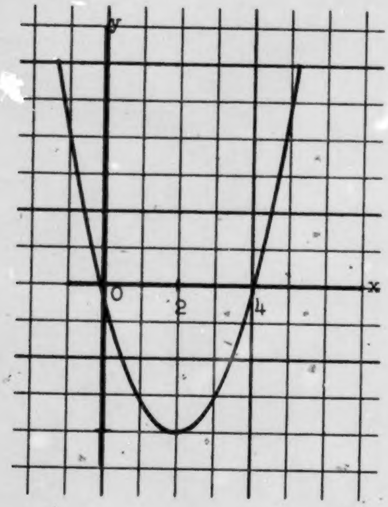
$g_1 : x \rightarrow \sqrt{4 - x^2}$ $g_2 : x \rightarrow -\sqrt{4 - x^2}$





12. Do Problem 11 for $f : x \rightarrow x^2 - 4x$.

(a)



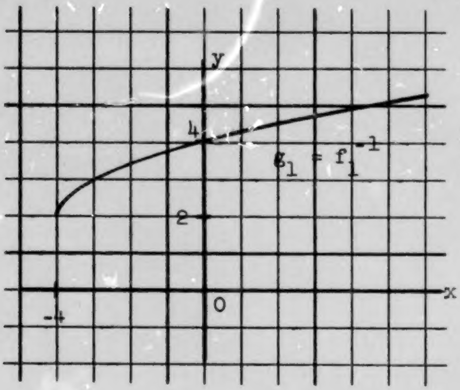
(b) $f_1 : x \rightarrow x^2 - 4x$

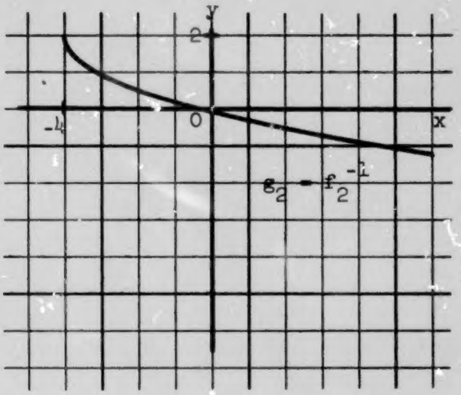
Domain $\{x : x \geq 2\}$

$f_2 : x \rightarrow x^2 - 4x$

Domain $\{x : x \leq 2\}$

(c) $g_1 : x \rightarrow 2 + \sqrt{4 + x}$ $\qquad$ $g_2 : x \rightarrow 2 - \sqrt{4 + x}$





13. Given that $f(x) = 3x - 2$ and $g(x) = -2x + k$, find k such that $fg(x) = gf(x)$. For this value of k are f and g inverse to one another? Give reasons for your answers.

No. $3(-2x + k) - 2 = -2(3x - 2) + k$ only for $k = 3$;

$$x \neq fg(x) = -6x + 7 = gf(x) = -6x + 7 \neq x .$$

The composition of two functions which are inverses is always the identity function, which is not the case here.

14. Show that $f : x \rightarrow x^2 - 4x + 5$ for $x \geq 2$ and $g : x \rightarrow 2 + \sqrt{x - 1}$ for $x \geq 1$ are inverse to one another by showing that $fg(y) = y$ for all y in the domain of g, and that $gf(x) = x$ for all x in the domain of f.

$$fg(y) = (2 + \sqrt{y - 1})^2 - 4(2 + \sqrt{y - 1}) + 5 = y , \ y \geq 1 ;$$

$$gf(x) = 2 + \sqrt{x^2 - 4x + 5 - 1} = 2 + \sqrt{(x - 2)^2} = 2 + |x - 2| = x , \ x \geq 2 .$$

15. If $f(x) = (2x^3 + 1)^7$, find at least two functions g such that $fg(x) = gf(x)$.

One solution for g will be the inverse function of f or

$g(x) = \sqrt[3]{\left(\frac{\sqrt[7]{x} - 1}{2}\right)}$. Other simple solutions are $g(x) = x$ and $g(x) = k$(constant) where $k = f(k) = (2k^3 + 1)^7$. This equation has at least one negative real root.

TC A2-4. Monotone Functions.

Example A2-4b. The comments (TC Example A2-1b) about the existence of a square root apply equally well to the existence of n-th roots. The same idea proves the existence of n-th roots. Thus, let A be the set of nonnegative numbers whose n-th power is less than p. Since $0 \in A$ and $p + 1$ is an upper bound for A, A has a least upper bound s. For $h < 1$

$$(s + h)^n = s^n + h \sum_{k=1}^{n} \binom{n}{k} s^{n-k} h^{k-1}$$

$$< s^n + h(n + 1)!s^n$$

$$< s^n + h(n + 1)!(p + 1)^n ,$$

and

$$(s - h)^n = s^h + \sum_{k=1}^{n} \binom{n}{k} s^{n-k} h^k (-1)^k$$

$$> s^h - h \sum_{\substack{k=1, \\ k \text{ odd}}}^{n} \binom{n}{k} s^{n-k} h^{k-1} (-1)^{k-1}$$

$$> s^n - h(n + 1)!(p + 1)^n .$$

Thus, as in TC Example A2-1b we can have neither $s^n < p$ nor $s^n > p$. Therefore $s^n = p$.

As stated in the text,

$$f : x \longrightarrow x^n , \ n \text{ an odd natural number,}$$

is a one-to-one function for all real x, and hence its inverse

$$g : y \longrightarrow \sqrt[n]{y} , \ n \text{ an odd natural number,}$$

is one-to-one for all real y. This raises notational problems. Since we write, for example, $\sqrt[3]{-27} = -3$, it is customary to write $\sqrt[3]{-y^3} = -y$ for all real y. In Chapter 8, however, fractional exponents are introduced, and we define $a^{1/n}$ only for $a > 0$. Thus we write $\sqrt[3]{y} = y^{1/3}$ only for $y > 0$.

One reason for this restriction is that it preserves the laws of exponents, in particular

$$(x^m)^n = x^{mn} .$$

Otherwise, the following ambiguity would arise:

$$[(-8)^{2/3}]^{1/2} = [-8]^{1/3} = -2 ,$$

and

$$[(-8)^{2/3}]^{1/2} = [4]^{1/2} = 2 .$$

There is an alternative way of defining the inverse of

$$f : x \rightarrow x^3 = y \text{ , for all real } x \text{ and } y ,$$

which you may wish to consider. If

$$x^3 = y ,$$

then

$$x = (\operatorname{sgn} y) \sqrt[3]{|y|} = \begin{cases} y^{1/3} & \text{if } y \geq 0 , \\ -|y|^{1/3} & \text{if } y < 0 . \end{cases}$$

For example, if $x^3 = -8$, then

$$x = (-1) \sqrt[3]{|-8|} = -|-8|^{1/3} = -2 .$$

We do not write $(-8)^{1/3}$.

Solutions Exercises A2-4

1. Prove that $f : x \rightarrow x^2$ for $x \geq 0$ is an increasing function. (Hint: Let $x_1 > x_2 \geq 0$; then $x_1 - x_2 > 0$. From this show that $x_1^2 > x_2^2$.)

$$x_1 > x_2 \geq 0 \Rightarrow x_1 + x_2 > 0 \text{ and } x_1 - x_2 > 0 \Rightarrow x_1^2 - x_2^2 > 0$$

$$\Rightarrow x_1^2 > x_2^2 .$$

2. Which of the following functions are nondecreasing? nonincreasing?[*] decreasing? increasing? In each case the domain is the set of real numbers unless otherwise restricted.

(a)	$f_1 : x \rightarrow c$, c a constant	Weakly increasing and weakly decreasing.
(b)	$f_2 : x \rightarrow x$	Increasing.
(c)	$f_3 : x \rightarrow \lvert x \rvert$	Not monotone.
(d)	$f_4 : x \rightarrow [x]$	Weakly increasing.
(e)	$f_5 : x \rightarrow \operatorname{sgn} x$	
(f)	$f_6 : x \rightarrow -x^2$, $x \leq 0$	Increasing.
(g)	$f_7 : x \rightarrow -\sqrt{x}$, $x \geq 0$	Decreasing.
(h)	$f_8 : x \rightarrow x\lvert x \rvert$	Increasing.
(i)	$f_9 : x \rightarrow x + \lvert x \rvert$	Weakly increasing.
(j)	$f_{10} : x \rightarrow \lvert x \rvert + \lvert x - 1 \rvert$	Not monotone.
(k)	$f_{11} : x \rightarrow \lvert x - 1 \rvert + \lvert x - 3 \rvert$	
(ℓ)	$g_1 : x \rightarrow f_3 f_4(x)$	
(m)	$g_2 : x \rightarrow f_4 f_3(x)$	

3. For each function in Problem 2 which is not monotone, divide its domain into parts such that the restriction of f to any of these parts gives a monotone or strongly monotone function.

(c) $x \leq 0$ and $x \geq 0$.

(j) $x \leq 0$ and $x \geq 0$ or $x \leq 1$ and $x \geq 1$.

(k) $x \leq 1$ and $x \geq 1$ or $x \leq 3$ and $x \geq 3$.

(ℓ) $x \leq 0$ and $x \geq 0$.

(m) $x \leq 0$ and $x \geq 0$.

---

[*] See footnote in A2-4.

4. We are given that the functions

f_1 is weakly increasing,
f_2 is increasing,
g_1 is weakly decreasing,
g_2 is decreasing,

in a common domain. What is the monotone character, if any, of the following functions:

(a)	$f_1 + f_2$,	Increasing.
(b)	$f_2 + g_1$,	Not necessarily monotone.
(c)	$g_1 + g_2$,	Decreasing.
(d)	$g_2 + f_1$,	Not necessarily monotone.
(e)	$f_1 \cdot f_2$,	
(f)	$f_2 \cdot g_1$,	
(g)	$g_1 \cdot g_2$,	
(h)	$g_2 \cdot f_1$,	
(i)	$f_1 f_2$,	Weakly increasing.
(j)	$f_2 f_1$,	
(k)	$f_2 g_1$,	Weakly decreasing.
(ℓ)	$g_1 f_2$,	
(m)	$g_1 g_2$,	Weakly increasing.
(n)	$g_2 g_1$,	
(o)	$g_2 f_1$,	Weakly decreasing.
(p)	$f_1 g_2$,	

TC A2-5. The Circular (Trigonometric) Functions.

There are three closely related, though distinct, kinds of trigonometric functions passing under the same name. First, there are the trigonometric functions of geometrical objects, namely, the angles introduced in geometry. Then, when we introduce an angle measure, the functions are functions of a real variable. The real functions depend upon the measure of the angles. Thus, the numerical functions obtained by measuring angles in degrees (a relic of the Babylonian sexagesimal numeration) are not the same as the functions obtained by measuring the angle in radians. Since an angle measured by x degrees is measured by $\frac{\pi}{180}x$ radians, a trigonometric function, say the sine function, defined in terms of degree measure is related to the corresponding trigonometric function defined in terms of radian measure by

$$\sin^{\circ} x = \sin \frac{\pi}{180} x .$$

Solutions Exercises A2-5

In the following problems you may assume properties 1 to 5 as given on page 304 . Also, after any particular identity has been proved it may be used as a true statement in any problem which follows.

1. Prove:

(a) $\sin 0 = 0$.

From (5) $\sin(-0) = -\sin 0$ or

$$2 \sin 0 = 0$$

(b) $\cos 0 = 1$.

$$1 = \sin \frac{\pi}{2} = \sin(\frac{\pi}{2} + 0)$$

$$= \sin \frac{\pi}{2} \cos 0 + \cos \frac{\pi}{2} \sin 0$$

$$= \cos 0$$

(c) $\cos \frac{\pi}{2} = 0$.

$$\sin^2 \frac{\pi}{2} + \cos^2 \frac{\pi}{2} = 1$$

$$\cos^2 \frac{\pi}{2} = 0$$

(d) $\sin \pi = 0$.

$$\sin \pi = \sin(\frac{\pi}{2} + \frac{\pi}{2})$$

$$= \sin \frac{\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2} \cos \frac{\pi}{2} = 0$$

(e) $\cos \pi = -1$. $\sin(\pi + \frac{\pi}{2}) = \cos \pi$ from (4) ;

$\sin(\pi + \frac{\pi}{2}) = \sin(-\frac{\pi}{2})$ from (1)

$= -1$

(f) $\cos(-\frac{\pi}{2}) = 0$. $\sin^2(-\frac{\pi}{2}) + \cos^2(-\frac{\pi}{2}) = 1$

2. Prove:

(a) $\sin(x + \frac{\pi}{2}) = \cos x$

$= \sin x \cos \frac{\pi}{2} + \cos x \sin \frac{\pi}{2}$

$= \sin x \cdot 0 + \cos x \cdot 1$

(b) $\sin(x + \pi) = -\sin x$

$= \sin x \cos \pi + \cos x \sin \pi$

$= \sin x(-1) + (\cos x) \cdot 0$

(c) $\cos(x + \frac{\pi}{2}) = -\sin x$

From (a) $\cos(x + \frac{\pi}{2}) = \sin(x + \pi)$

$= \sin x \cos \pi + \cos x \sin \pi = -\sin x$.

(d) $\cos(x + \pi) = -\cos x$

From (c) $\cos(x + \pi) = -\sin(x + \frac{\pi}{2})$

$= -(\sin x \cos \frac{\pi}{2} + \sin \frac{\pi}{2} \cos x) = -\cos x$.

(e) $\cos x = \cos(-x)$ or equivalently $\cos(x - \frac{\pi}{2}) = \cos(\frac{\pi}{2} - x)$

$-\sin(x - \frac{\pi}{2}) = \sin(\frac{\pi}{2} - x) = \sin \frac{\pi}{2} \cos(-x) + \sin(-x)\cos \frac{\pi}{2} = \cos(-x)$

$= \cos x$ from (c) .

3. Prove:

(a) $\sin(x - y) = \sin x \cos y - \cos x \sin y$

Use (4), (5) and Number 2(e).

(b) $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\cos(x \pm y) = \sin(x \pm y + \frac{\pi}{2}) = \sin([x + \frac{\pi}{2}] \pm y)$,
use (4), (5), Number 2(a), (c) and 3(a).

(c) $\sin 2t = 2 \sin t \cos t$

$t = x = y$ in (4).

(d) $\cos 2t = \cos^2 t - \sin^2 t$

$t = x = y$ in (b) of this exercise.

(e) $\sin 3t = 3 \sin t - 4 \sin^3 t$

Use Numbers 3(c), 3(d) and Property (4).

(f) $\cos 3t = 4 \cos^3 t - 3 \cos t$

Use Numbers 2(a) and 3(e).

(g) $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

Use $\tan = \frac{\sin}{\cos}$ and divide by $\cos x \cos y$.

(h) $\sin p - \sin q = 2 \cos \frac{p + q}{2} \sin \frac{p - q}{2}$

Use $\sin(x \pm y)$, letting $x = \frac{p + q}{2}$ and $y = \frac{p - q}{2}$.

(i) $\cos p - \cos q = -2 \sin \frac{p + q}{2} \sin \frac{p - q}{2}$

Use $\cos(x \pm y)$ as in Number 3(h).

(j) $\tan \frac{t}{2} = \frac{1 - \cos t}{\sin t} = \frac{\sin t}{1 + \cos t}$

Divide 3(i) by 3(h), letting $q = 0$.

4. Prove:

(a) $\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}}$

(b) $\cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}}$

(c) Explain the significance of the ± sign in (a) and (b).

For (a) and (b) use (d) of Number 3, letting $x = 2t$ and $t = \frac{x}{2}$.

The significance of the ± sign is that the square root may be either positive or negative. Apparently which sign to use depends on the value of x. To simplify the discussion, we will also assume that $\sin x > 0$ for $0 < x < \frac{\pi}{2}$. It now follows that

$$\sin x > 0 \quad \text{for} \quad 0 < x < \pi ,$$

$$\sin x < 0 \quad \text{for} \quad \pi < x < 2\pi ,$$

from

$$\sin x = \sin(\pi - x)$$

and

$$-\sin x = \sin(\pi + x) .$$

Furthermore, from $\cos x = \sin(x + \frac{\pi}{2})$, it then follows that

$$\cos x > 0 \quad \text{for} \quad 0 < x < \frac{\pi}{2}$$

and

$$\frac{3\pi}{2} < x < 2\pi ,$$

and

$$\cos x < 0 \quad \text{for} \quad \frac{\pi}{2} < x < \frac{3\pi}{2} .$$

5. Determine the numerical values of the following:

(a) $\sin \frac{3\pi}{2} = \sin(\pi + \frac{\pi}{2}) = - \sin \frac{\pi}{2} = -1$

(b) $\cos \frac{13\pi}{6} = \cos(2\pi + \frac{\pi}{6}) = \cos \frac{\pi}{6}$.

Since $0 = \cos \frac{\pi}{2} = \cos 3(\frac{\pi}{6}) = 4 \cos^3(\frac{\pi}{6}) - 3 \cos \frac{\pi}{6}$, $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$:

(c) $\tan\frac{5\pi}{3} = \dfrac{\sin\frac{5\pi}{3}}{\cos\frac{5\pi}{3}} = \dfrac{-\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}}$

Since $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $\cos\frac{\pi}{3} = 2(\frac{3}{4}) - 1 = \frac{1}{2}$.

Since, $\sin^2 x + \cos^2 x = 1$, $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

$$\tan\frac{5\pi}{3} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}\ .$$

(d) $\sin(-\frac{\pi}{4}) = -\sin\frac{\pi}{4} = -\sqrt{\dfrac{1-\cos\frac{\pi}{2}}{2}} = -\sqrt{\frac{1}{2}} = -\frac{\sqrt{2}}{2}$.

(e) $\sin(\frac{5\pi}{12}) = \sin(\frac{\pi}{4} + \frac{\pi}{6}) = \sin\frac{\pi}{4}\cdot\cos\frac{\pi}{6} + \cos\frac{\pi}{4}\cdot\sin\frac{\pi}{6}$

$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $\sin\frac{\pi}{6} = \frac{1}{2}$

$\sin(\frac{5\pi}{12}) = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.

(f) $\tan\frac{\pi}{6} = \dfrac{\sin\frac{\pi}{6}}{\cos\frac{\pi}{6}} = \dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \dfrac{\sqrt{3}}{3}$

(g) $\cos\frac{5\pi}{8} = \cos(\frac{\pi}{2} + \frac{\pi}{8}) = -\sin\frac{\pi}{8} = -\dfrac{\sqrt{2-\sqrt{2}}}{2}$

$\sin\frac{\pi}{8} = \sqrt{\dfrac{1-\cos\frac{\pi}{4}}{2}} = \sqrt{\dfrac{1-\frac{\sqrt{2}}{2}}{2}} = \dfrac{\sqrt{2-\sqrt{2}}}{2}$

(h) $\sin\frac{\pi}{16} = \sqrt{\dfrac{1-\cos\frac{\pi}{8}}{2}}$.

Since $\cos\frac{\pi}{8} = \sqrt{\dfrac{1+\frac{\sqrt{2}}{2}}{2}} = \dfrac{\sqrt{2+\sqrt{2}}}{2}$,

$$\sin\frac{\pi}{16} = \sqrt{\frac{1-\frac{\sqrt{2+\sqrt{2}}}{2}}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}\ .$$

(i) $\cos^3 (\frac{3\pi}{8}) = [\cos (\frac{\pi}{2} - \frac{\pi}{8})]^3 = [-\sin (-\frac{\pi}{8})]^3 = [\sin \frac{\pi}{8}]^3 = \left(\frac{\sqrt{2 - \sqrt{2}}}{2}\right)^3$

Λ (j) $\tan \frac{\pi}{5} = \sqrt{5 - 2\sqrt{5}}$

$\sin(\frac{\pi}{2} - x) = \cos x$, $\sin 2(\frac{\pi}{10}) = \cos 3(\frac{\pi}{10})$,

$2 \sin \frac{\pi}{10} \cos \frac{\pi}{10} = 4 \cos^3 \frac{\pi}{10} - 3 \cos \frac{\pi}{10}$.

$\cos \frac{\pi}{10} > 0$, $4 \cos^2 \frac{\pi}{10} - 2 \sin \frac{\pi}{10} = 3$,

$4 \sin^2 \frac{\pi}{10} + 2 \sin \frac{\pi}{10} - 1 = 0$.

Thus, $\sin \frac{\pi}{10} = \frac{-1 + \sqrt{5}}{4}$,

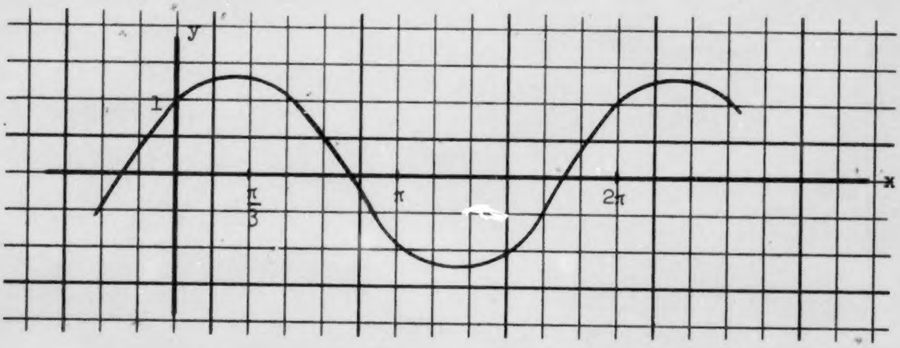
$\cos \frac{\pi}{5} = \cos 2(\frac{\pi}{10}) = 1 - 2 \sin^2 \frac{\pi}{10} = \frac{1 + \sqrt{5}}{4}$,

and $\sin \frac{\pi}{5} = \sqrt{1 - \cos^2 \frac{\pi}{5}} = \frac{\sqrt{10 - 2\sqrt{5}}}{4}$.

Finally, $\tan \frac{\pi}{5} = \frac{\sqrt{10 - 2\sqrt{5}}}{1 + \sqrt{5}} = \sqrt{5 - 2\sqrt{5}} = .727$.

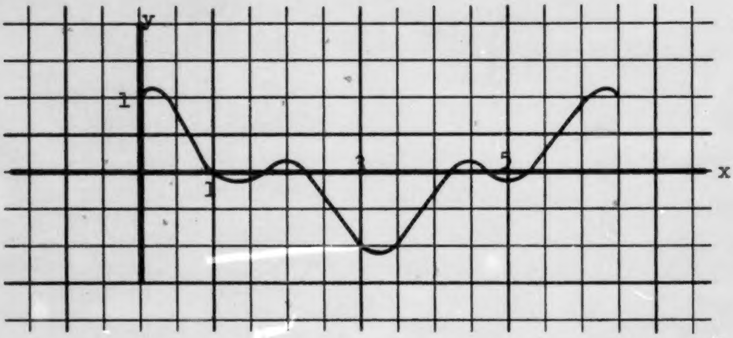
6. Sketch the graph of each of the following functions. For those functions which are periodic, indicate their periods. Indicate those functions which are even or odd.

(a) $f : x \to \sin x + \cos x$ Period $= 2\pi$

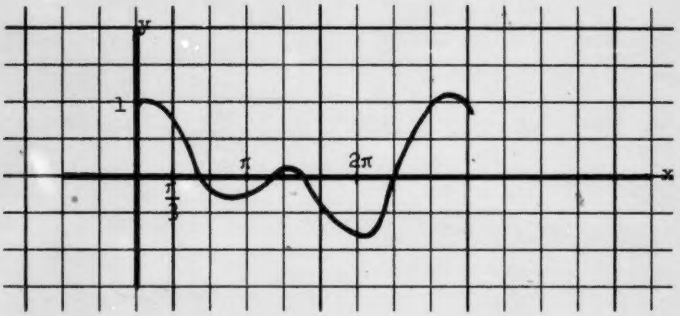


Also note: $f(x) = \sqrt{2} \sin (x + \frac{\pi}{4})$.

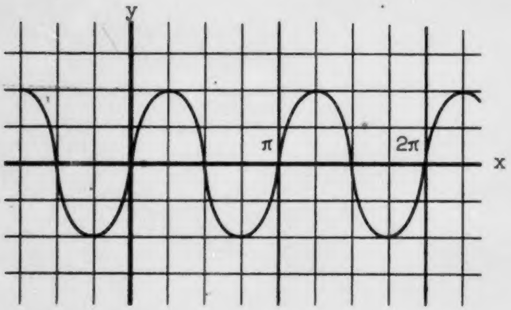
(b) $f : x \to \sin \frac{2\pi}{3} x + \cos \pi x$ Period = 6



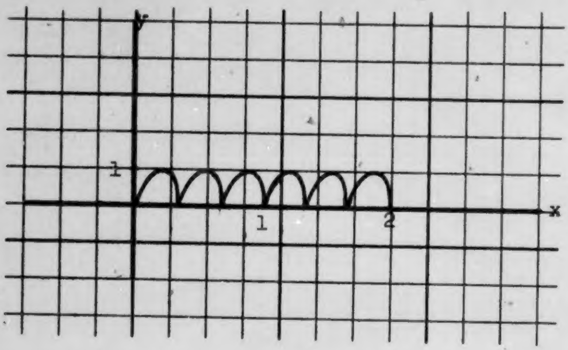
(c) $f : x \to \sin x + \cos x\sqrt{2}$



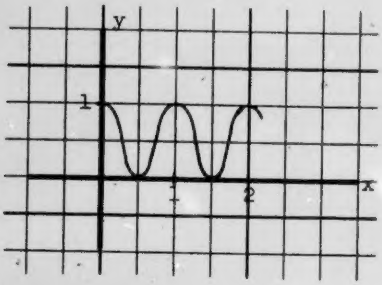
(d) $f : x \to \sin 2x$ Period = π . odd.



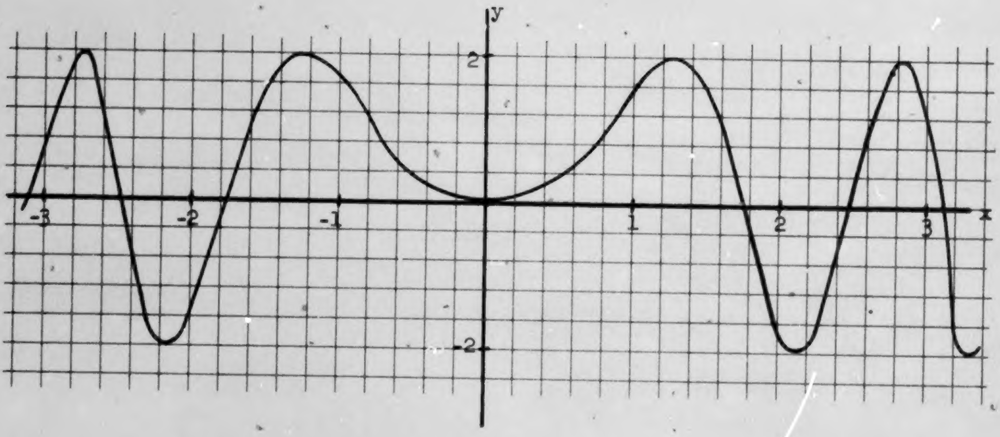
(e) f : x ⟶ $|\sin 3\pi x|$ Period = $\frac{1}{3}$. even.



(f) f : x ⟶ $\cos^2 \pi x$ Period = 1 . even.



(g) f : x ⟶ $2 \sin x^2$ even.



7. (a) What is the period of $\sin ax$, $a \neq 0$?

   (b) What is the period of $\sin(ax + b)$, $a \neq 0$?

   (c) For what values of a and b is the function odd? even?

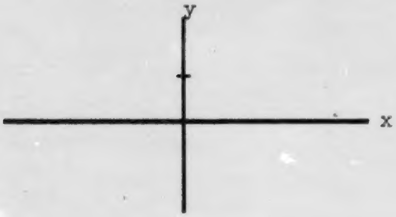
In (a) and (b), the period is $\frac{2\pi}{a}$. For $a \neq 0$, $b = n\pi$, n an integer, then $x \to \sin(ax + b)$ is an odd function.

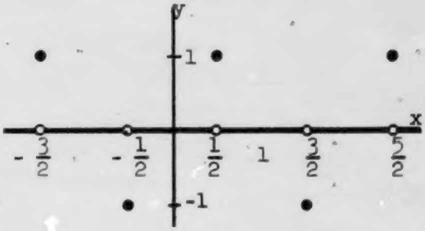
For $b = \frac{2n + 1}{2}\pi$, n an integer, $x \to \sin(ax + b)$ is an even function.

8. For each pair of functions f and g find the composite functions fg and gf and specify the domain and range, (if possible) of each. Also, sketch the graph of each, and give the period (fundamental) of those which are periodic.

   (a) $f : x \to \sin \pi x$; $g : x \to \operatorname{sgn} x$.

$fg : x \to \sin \pi(\operatorname{sgn} x) = 0$. $gf : x \to \operatorname{sgn}(\sin \pi x)$





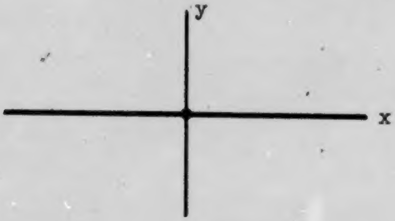
Domain: all reals.
Range: {0} .
Period = 0 .

Domain: all reals.
Range: {-1,0,1} .
period = 2 .

(b) $f : x \rightarrow \sin a\pi x$, $a > 0$; $g : x \rightarrow \operatorname{sgn} x$

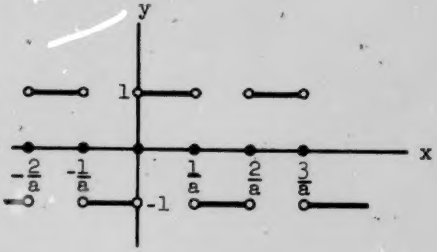
$fg : x \rightarrow \sin a\pi(\operatorname{sgn} x)$; $gf(x) = \operatorname{sgn}(\sin a\pi x)$

if a is an integer:



Domain: all reals.

Range: $\{0\}$.

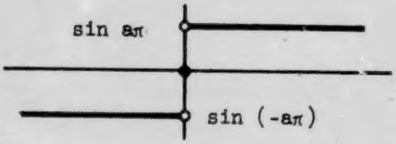


Domain: all real numbers.

Range: $\{-1,0,1\}$

Period $= \frac{2}{a}$.

If $a \neq$ an integer:

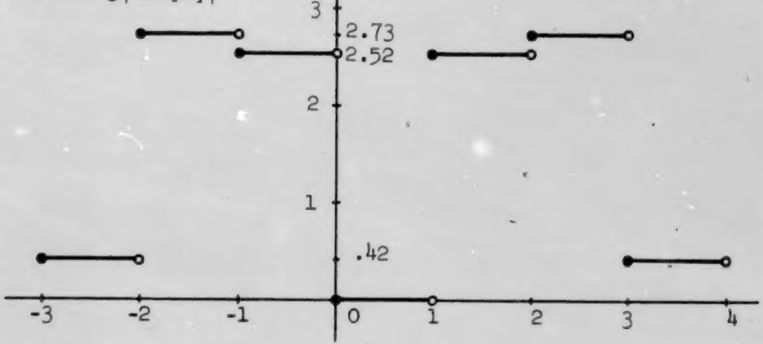


Domain: all reals.

Range: $\{\sin(-a\pi) , 0 , \sin a\pi\}$

(c) $f : x \rightarrow 3|\sin x|$; $g(x) = [x]$

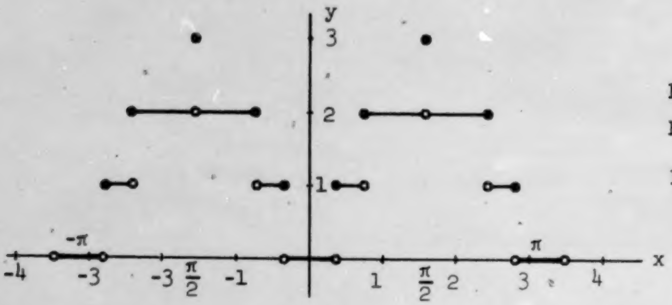
$fg : x \rightarrow 3|\sin[x]|$



Domain: all reals

Range: $3|\sin n|$, n integer

$gf : x \rightarrow [3|\sin x|]$



Domain: all reals.

Range: 0 , 1 , 2 , 3

Period = π .

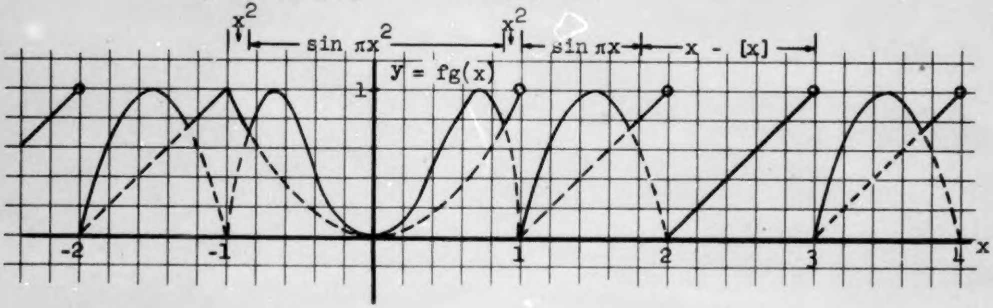
∧(d) $f : x \rightarrow \max\{\sin \pi x , x - [x]\}$; $g : x \rightarrow \min\{x , x^2\}$.

$fg : x \rightarrow \max\{\sin \pi(\min\{x,x^2\}) , (\min\{x,x^2\} - [\min\{x,x^2\}])\}$

For $|x| < 1$, $x^2 < x$. So, fg can be written as

$$fg(x) = \begin{cases} \max\{\sin \pi x , x - [x]\} , & |x| > 1 \\ \max\{\sin \pi x^2 , x^2 - [x^2]\} , & |x| < 1 \end{cases}$$

The graph of fg is given by the solid line curve. The value of fg , in the various intervals, is given by the functions indicated on top of the graph in that interval.



Periodic outside of the interval [-1,1] with period = 2 .

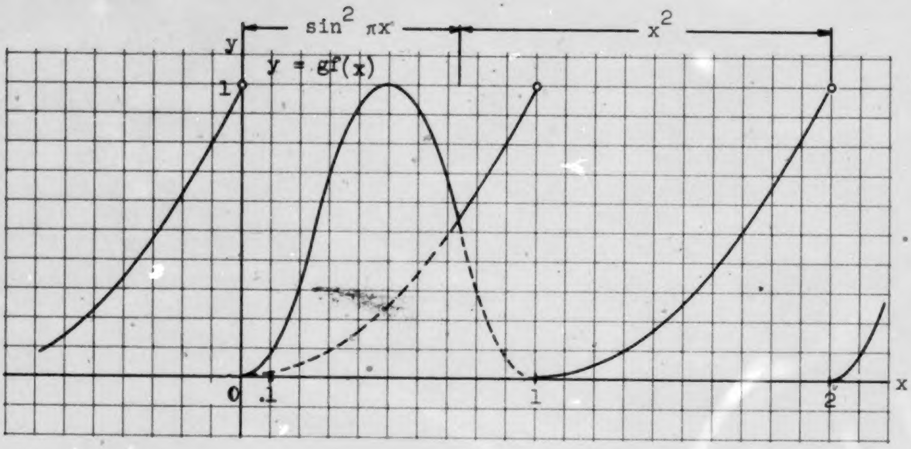
$gf : x \rightarrow \min\{(\max\{\sin \pi x , x - [x]\} , \max^2\{\sin \pi x , x - [x]\}$.

Since, $\max\{\sin \pi x , x - [x]\}$ is always between 0 and 1 , the $\max^2$ will be the minimum.

We may therefore write

$$gf : x \rightarrow (\max\{\sin \pi x , x - [x]\})^2 .$$

This is periodic with period = 2 .



9. Solve for all x in $[0,2\pi]$:

$$\sin^m x + \cos^m x = 1 , \qquad (m \text{ an integer } > 2)$$

Since, $\sin^2 x + \cos^2 x = 1$ for all x , and

$$\sin^2 x \geq \sin^m x \quad ,$$

$$\cos^2 x \geq \cos^m x \quad ,$$

solutions must satisfy

$$\sin^m x = 1 \quad \text{or} \quad \cos^m x = 1 .$$

For m , even, $x = \frac{\pi}{2} , \frac{3\pi}{2}$ or $x = 0 , \pi$.

For m , odd, $x = \frac{\pi}{2}$ or $x = 0$.

$\therefore$ m odd has as solutions $x = 0 , \frac{\pi}{2}$.

For m even, solutions are $x = 0 , \frac{\pi}{2} , \pi , \frac{3\pi}{2}$.

Λ10. If (3) on page 304 is replaced by $\cos x = \sin(x + \frac{\pi}{2})$, then using (1) to (5), show that

$$\cos^2 x + \sin^2 x = 1 .$$

A notation in parentheses after a statement will indicate the basis for the step. The first five statements are the axioms. The derived statements are numbered in sequence starting from 6 .

(1) $\sin x$ is periodic with period 2π .

(2) $\sin \frac{\pi}{2} = 1$.

(3) $\sin(x + \frac{\pi}{2}) = \cos x$.

(4) $\sin(x + y) = \sin x \cos y + \cos x \sin y$.

(5) $\sin(-x) = -\sin x$.

(6) $\sin 0 = 0$, (5) .

(7) $\cos 0 = 1$, (3) .

(8) $\cos(-\frac{\pi}{2}) = 0$, (3) , (6) .

(9) $\sin(x - \frac{\pi}{2}) = \sin x \cos(-\frac{\pi}{2}) + \sin(-\frac{\pi}{2}) \cos x = -\cos x$, (4) , (8) , (2) , (5) .

(10) $\cos(\frac{\pi}{2}) = 0$, (9) , (6) .

(11) $\sin \pi = 0$, (3) , (10) .

(12) $\sin(\frac{\pi}{2} - x) = \sin \frac{\pi}{2} \cos(-x) + \sin(-x) \cos \frac{\pi}{2} = \cos(-x)$, (4) , (2) , (10)

(13) $\cos x = \cos(-x)$, (5) , (9) , (12) .

(14) $\cos(x + \frac{\pi}{2}) = \sin(-x) = -\sin x$, (9) , (5) .

(15) $\sin(x + y + \frac{\pi}{2}) = \cos(x + y) = \sin(x + \frac{\pi}{2}) \cos y + \sin y \cos(x + \frac{\pi}{2})$, (3) , (4) .

(16) $\cos(x + y) = \cos x \cos y - \sin x \sin y$, (15) , (3) , (14) .

(17) $\sin^2(x + y) + \cos^2(x + y) = (\cos^2 x + \sin^2 x)(\cos^2 y + \sin^2 y)$, (4) , (16) .

If $y = -x$, (17) becomes: $\sin^2 0 + \cos^2 0 = 1 = (\cos^2 x + \sin^2 x)^2$, (6) , (7) .

And so, $\cos^2 x + \sin^2 x = 1$.

Λ 11. Show how to solve the cubic equation $4x^3 - 3x = a$ $(|a| \leq 1)$ trigonometrically.

Let $x = \cos \theta$, $4 \cos^3 \theta - 3 \cos \theta = \cos 3\theta$ by Number 3(f). So, $\cos 3\theta = a$; find 3θ from trigonometric tables and then $\cos \theta = x$.

TC A2-6. Polar Coordinates.

Descartes introduced oblique axes as well as perpendicular axes. The only reason that rectangular coordinate axes are preferred to oblique axes (at an angle θ, $0 < \theta < \frac{\pi}{2}$) is that the formula expressing the distance between points would become more complicated (by involving the familiar Law of Cosines).

Solutions Exercises A2-6

1. Find all polar coordinates of each of the following points:

(a) $(6,\frac{\pi}{4}) = (6,\frac{\pi}{4} + 2n\pi)$, $\left(-6,\frac{\pi}{4} + (2n + 1)\pi\right)$.

(b) $(-6,\frac{\pi}{4}) = (-6,\frac{\pi}{4} + 2n\pi)$, $\left(6,\frac{\pi}{4} + (2n + 1)\pi\right)$.

(c) $(6,-\frac{\pi}{4}) = (6,-\frac{\pi}{4} + 2n\pi)$, $\left(-6,-\frac{\pi}{4} + (2n + 1)\pi\right)$.

(d) $(-6,-\frac{\pi}{4}) = (-6,-\frac{\pi}{4} + 2n\pi)$, $\left(6,-\frac{\pi}{4} + (2n + 1)\pi\right)$.

2. Find rectangular coordinates of the points in Number 1.

(a) $(6,\frac{\pi}{4}) = (3\sqrt{2},3\sqrt{2})$

(b) $(-6,\frac{\pi}{4}) = (-3\sqrt{2},-3\sqrt{2})$

(c) $(6,-\frac{\pi}{4}) = (3\sqrt{2},-3\sqrt{2})$

(d) $(-6,-\frac{\pi}{4}) = (-3\sqrt{2},3\sqrt{2})$

3. Find polar coordinates of each of the following points given in rectangular coordinates:

(a) $(4,-4) = (4\sqrt{2},-\frac{\pi}{4} + 2n\pi)$

(b) $(-\frac{3\sqrt{3}}{2},\frac{3}{2}) = (3,\frac{5\pi}{6} + 2n\pi)$

(c) $(-2,-2\sqrt{3}) = (4,\frac{4\pi}{3} + 2n\pi)$

(d) $(0,-10) = (10,\frac{3\pi}{2} + 2n\pi)$

(e) $(-3,0) = \left(3,(2n + 1)\pi\right)$

(f) $(-3,4) = (-5, \arcsin \frac{-4}{5} + 2n\pi)$ or $\left(-5, (\arctan -\frac{4}{3}) + 2n\pi\right)$

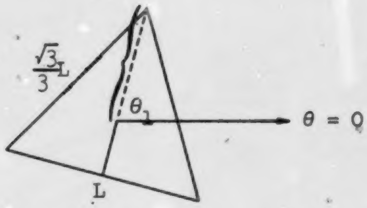
(g) $(-\sqrt{3}, 1) = (2, \frac{5\pi}{6} + 2n\pi)$

(h) $(\sqrt{2}, -\sqrt{2}) = (2, \frac{7\pi}{4} + 2n\pi)$

4. Given the cartesian coordinates (x,y) of a point, formulate unique polar coordinates (r,θ) for $0 \leq \theta \leq \pi$. (Hint: use $\arccos \frac{x}{r}$.)

$$\left(\frac{y}{|y|}\sqrt{x^2 + y^2}, \arccos \frac{x}{\sqrt{x^2 + y^2}}\right)$$

5. Determine the polar coordinates of the 3 vertices of an equilateral triangle if a side of the triangle has length L, the centroid of the triangle coincides with the pole, and one angular coordinate of a vertex is θ_1 radians.



The centroid is at the intersection of the medians.

$\frac{2}{3} \cdot L \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{3} L$ = distance of vertex from centroid.

Coordinates of the 3 vertices are:

$(\frac{\sqrt{3}}{3} L, \theta_1)$, $(\frac{\sqrt{3}}{3} L, \theta_1 + \frac{2\pi}{3})$, $(\frac{\sqrt{3}}{3} L, \theta_1 + \frac{4\pi}{3})$.

6. Find equations, in polar coordinates, of the following curves:

(a)	$x = c$, c a constant.	$r \cos \theta = c$
(b)	$y = c$, c a constant.	$r \sin \theta = c$
(c)	$ax + by = c$.	$r(a \cos \theta + b \sin \theta) = c$
(d)	$x^2 + (y - k)^2 = k^2$.	$r = 2k \sin \theta$
(e)	$y^2 = 4ax$.	$r^2 \sin^2 \theta = r \cdot 4a \cos \theta$
		or $r = \frac{4a \cos \theta}{1 - \cos^2 \theta}$
(f)	$x^2 - y^2 = a^2$.	$r^2 \cos 2\theta = a^2$

7. Find equations, in rectangular coordinates, of the following curves:

(a) $r = a$. $x^2 + y^2 = a^2$

(b) $r \sin\theta = -5$. $y = -5$

(c) $r = 2a \sin\theta$. $x^2 + (y - a)^2 = a^2$. See Number 5(d).

(d) $r = \frac{1}{1 - \cos\theta}$.

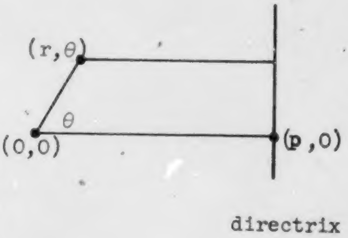
$$\pm\sqrt{x^2 + y^2} - x = 1 ,$$
$$x^2 + y^2 = x^2 + 2x + 1 ,$$
$$y^2 = 2x + 1 , \quad \text{or}$$
$$x = \frac{1}{2}y^2 - \frac{1}{2} .$$

(e) $r = 2 \tan\theta$.

$$x^2 + y^2 = \frac{4y^2}{x^2} \quad \text{or}$$
$$x^4 + x^2y^2 - 4y^2 = 0 .$$

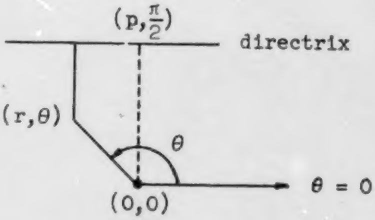
8. Derive an equation in polar coordinates for conic sections with a focus at the pole and directrix perpendicular to the polar axis and p units to the right of the pole.

$$\frac{r}{p - r\cos\theta} = \epsilon .$$
$$r(1 + \epsilon\cos\theta) = \epsilon p$$
$$r = \frac{\epsilon p}{1 + \epsilon\cos\theta}$$



9. Repeat Number 8 if the directrix is parallel to the polar axis and p units above the focus at the pole.

$$\frac{r}{p - r\sin\theta} = \epsilon$$
$$r = \frac{\epsilon p}{1 + \epsilon\sin\theta}$$



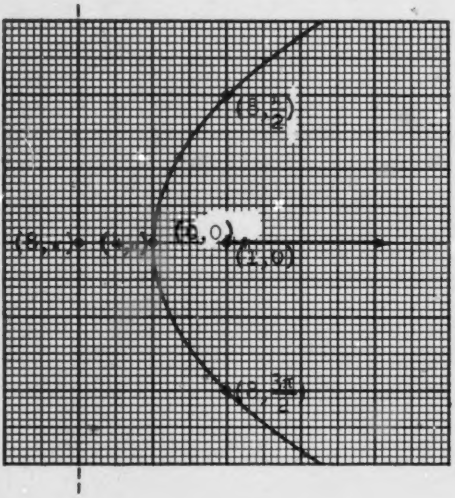
10. Repeat Number 8 if the directrix is parallel to the polar axis and p units below the focus at the pole.

$$r = \frac{\epsilon p}{1 - \epsilon \sin \theta}$$

11. Discuss and sketch each of the following curves in polar coordinates:

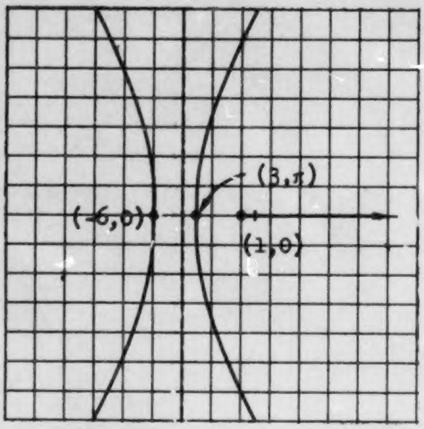
(a) $r = \dfrac{8}{1 - \cos \theta}$

Since $\epsilon = 1$, the graph is a parabola and its directrix is 8 units to the left of the pole.



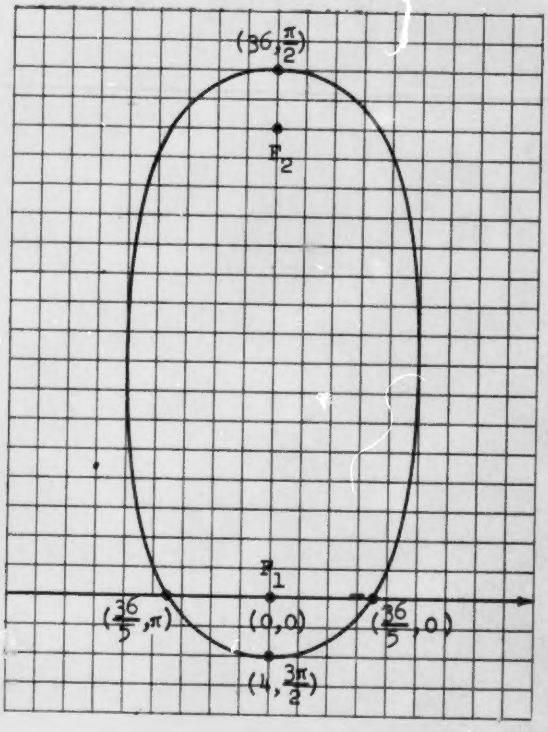
(b) $r = \dfrac{12}{1 - 3 \cos \theta}$

Since $\epsilon = 3$, the graph is a hyperbola with its directrix 4 units to the left of the pole.



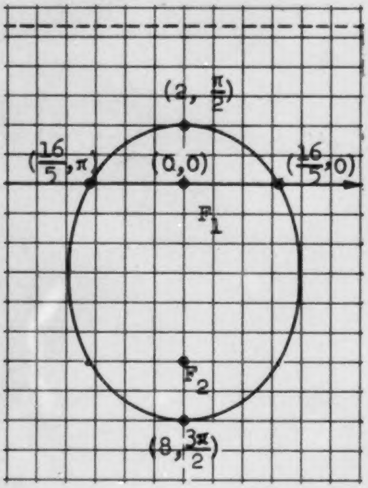
(c) $r = \dfrac{36}{5 - 4 \sin \theta} = \dfrac{\frac{36}{5}}{1 - \frac{4}{5} \sin \theta}$

Since $\epsilon = \frac{4}{5}$, the graph is an ellipse with its directrix 9 units below the pole.



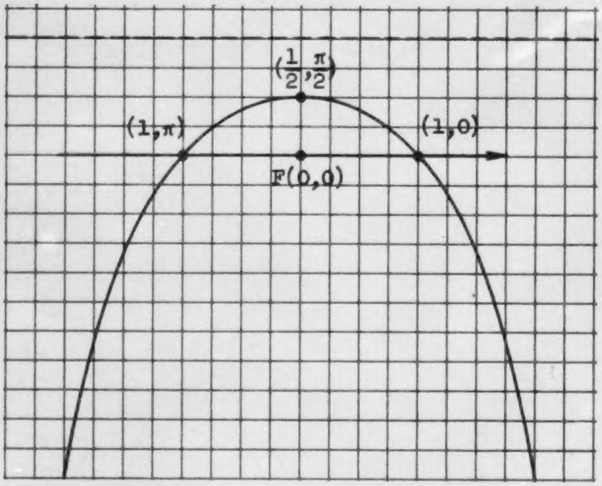
(d) $r = \dfrac{16}{5 + 3 \sin \theta} = \dfrac{\frac{16}{5}}{1 + \frac{3}{5} \sin \theta}$

Since $\epsilon = \frac{3}{5}$, the graph is an ellipse with its directrix $\frac{16}{3}$ units above the pole.



(e) $r \sin \theta = 1 - r$ or $r = \dfrac{1}{1 + \sin \theta}$

Since $\epsilon = 1$, the graph is a parabola with its directrix one unit above the pole.



12. Certain types of symmetry of curves in polar coordinates are readily detected. For example, a curve is symmetric about the pole if the equation is unchanged when r is replaced by $-r$. What kind of symmetry occurs if an equation is unchanged when

(a) θ is replaced by $-\theta$?

Symmetric about the polar axis.

(b) θ is replaced by $\pi - \theta$?

Symmetric about the line $\theta = \frac{\pi}{2}$.

(c) r and θ are replaced by $-r$ and $-\theta$, respectively?

Symmetric about the line $\theta = \frac{\pi}{2}$.

(d) θ is replaced by $\pi + \theta$?

Symmetric about the pole.

13. Without actually sketching the graphs, describe the symmetries of the graphs of the following equations:

(a) $r^2 = 4 \sin 2\theta$.

Symmetric about the pole. $f(\theta) = f(\theta + \pi)$.

Two lines of symmetry can be demonstrated,

the lines, $\theta = \frac{\pi}{4}$ and $-\frac{\pi}{4}$.

$4 \sin 2(\frac{\pi}{4} - \theta) = 4 \sin 2(\frac{\pi}{4} + \theta)$,

(Earlier, it was shown $\sin(\frac{\pi}{2} + x) = \sin(\frac{\pi}{2} - x)$),

and $4 \sin 2(-\frac{\pi}{4} + \theta) = 4 \sin 2(-\frac{\pi}{4} - \theta)$,

since $\sin(-\frac{\pi}{2} + \theta) = -\sin(\frac{\pi}{2} - \theta)$.

(b) $r(1 - \cos \theta) = 10$

Since $f(\theta) = f(-\theta)$, symmetric about the polar axis.

(c) $r = \cos^2 2\theta$

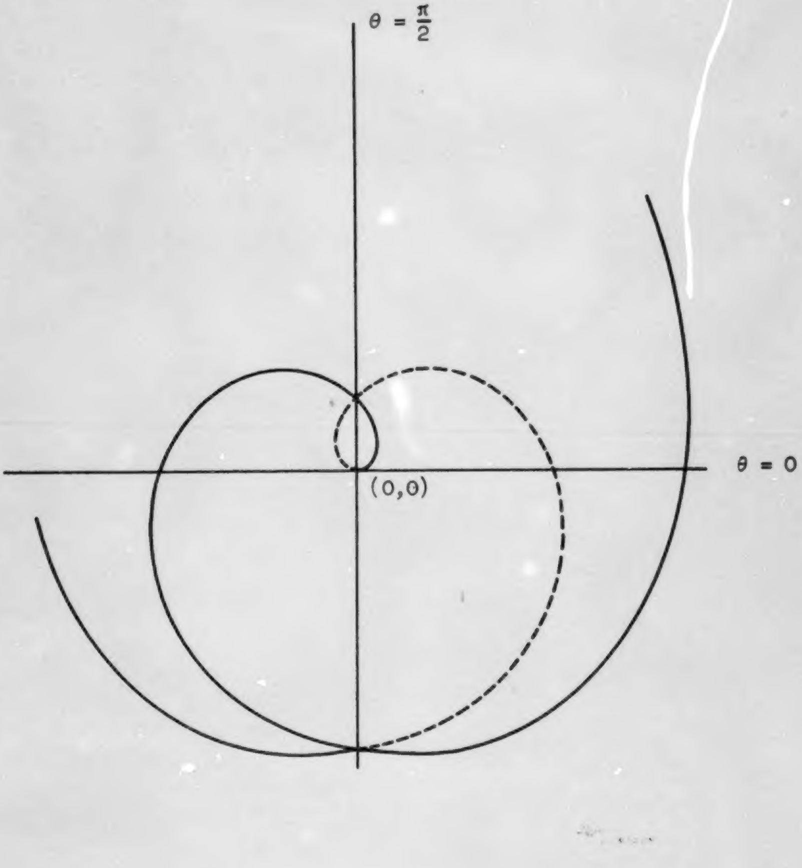
Since $f(\theta) = f(-\theta)$, symmetric about polar axis.

Since $f(\theta) = f(\pi - \theta)$, it is symmetric about the line $\theta = \frac{\pi}{2}$.

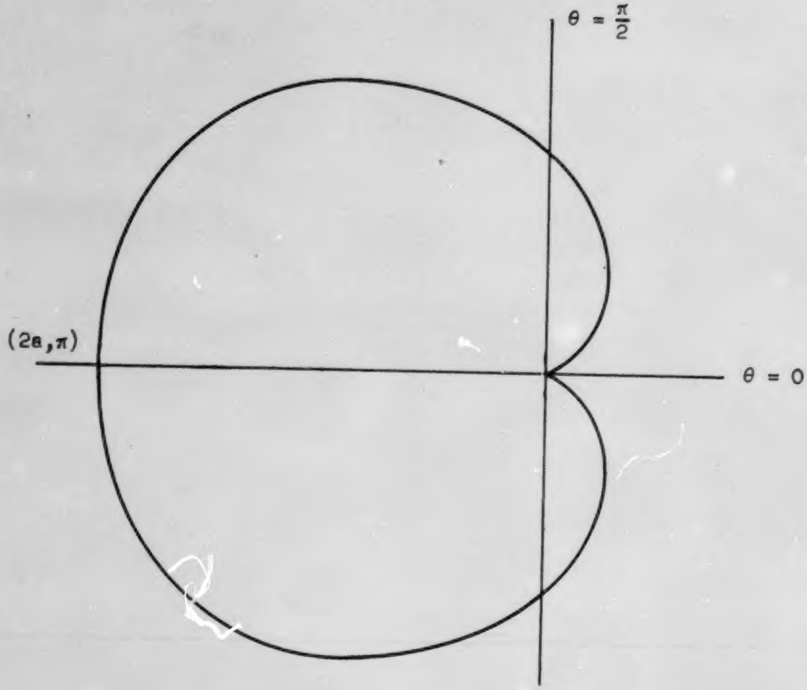
Since $\cos(x + \pi) = -\cos x$, and this angle is 2θ , and the cos is squared, the lines $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$ are axes of symmetry.

14. Sketch the following curves in polar coordinates:

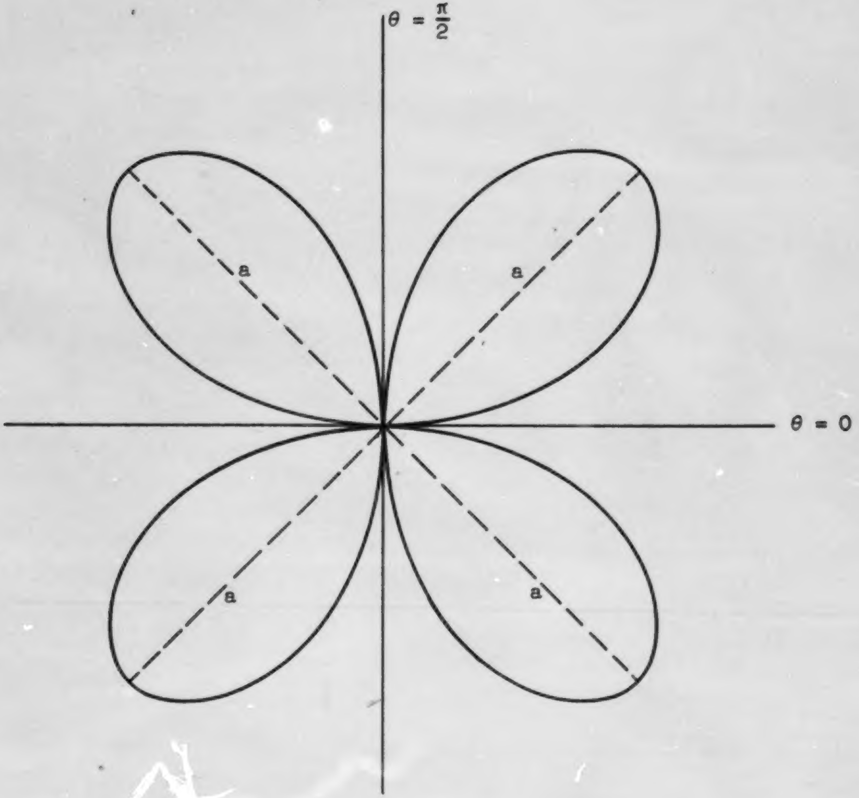
(a) $r = a\theta$



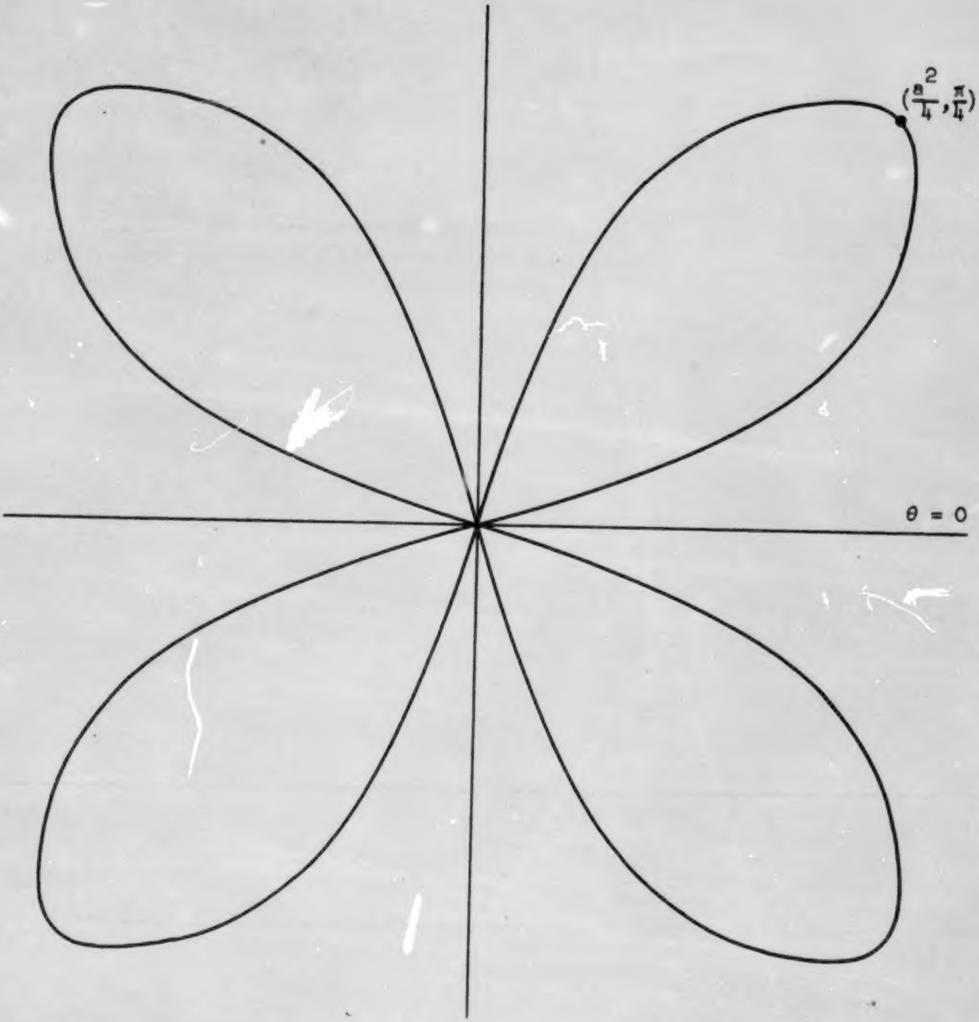
(b) $r = a(1 - \cos \theta)$



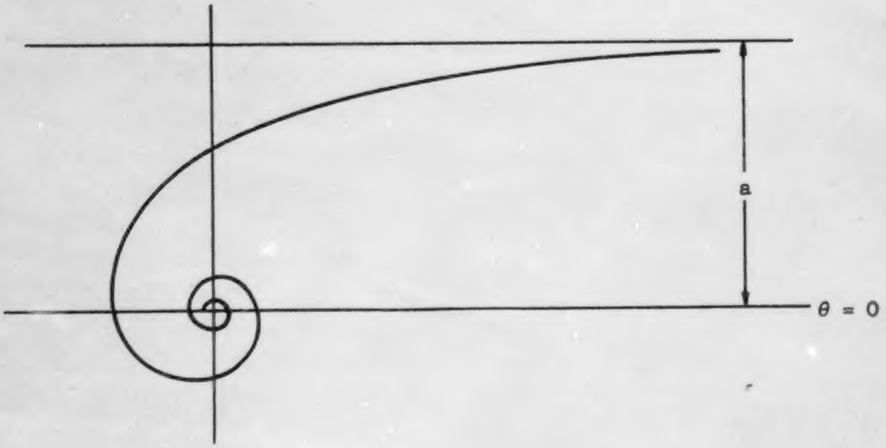
(c) $r = a \sin 2\theta$


$\theta = \frac{\pi}{2}$
$\theta = 0$
a
a
a
a

(d) $r = a^2 \sin^2 \theta \cos^2 \theta$



(e) $r\theta = a$



15. In each of the following, find all points of intersection of the given pairs of equations. (Recall that the polar representation of a point is not unique.)

(a) $r = 2 - 2 \sin \theta$, $r = 2 - 2 \cos \theta$.

$\sin \theta = \cos \theta$, $\tan \theta = 1$

$\theta = \frac{\pi}{4}$, $\frac{5\pi}{4}$

Points are $(2 - \sqrt{2}, \frac{\pi}{4})$ and $(2 + \sqrt{2}, \frac{5\pi}{4})$ and since r in each equation may equal 0 , the point $(0,\theta)$ is common to both (for different θ).

(b) $r = -2 \sin 2\theta$, $r = 2 \cos \theta$.

$-2 \sin 2\theta = -4 \sin \theta \cos \theta = 2 \cos \theta$

$\sin \theta = -\frac{1}{2}$ or $\cos \theta = 0$

$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$ or $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$(0,\frac{\pi}{2}) = (0,\frac{3\pi}{2})$ and $(-\sqrt{3}, \frac{7\pi}{6})$, $(\sqrt{3}, \frac{11\pi}{6})$

(c) $r = 4(1 + \cos \theta)$, $r(1 - \cos \theta) = 3$

$4(1 + \cos \theta)(1 - \cos \theta) = 3$

$1 - \cos^2 \theta = \frac{3}{4}$

$\sin^2 \theta = \frac{3}{4}$, $\sin \theta = \pm \frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

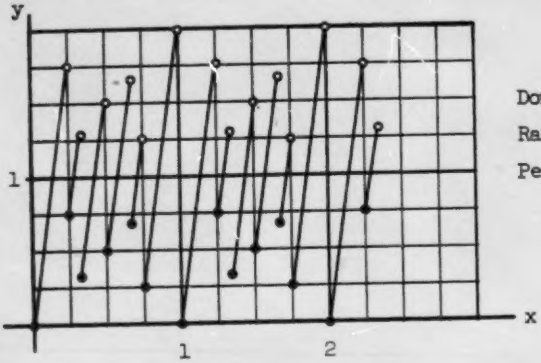
$(6,\frac{\pi}{3})$, $(2, \frac{2\pi}{3})$, $(2, \frac{4\pi}{3})$, $(6, \frac{5\pi}{3})$

Suggested Test Items and Solutions Appendix 2

The items included in this collection were selected because they cover ideas pertinent to the calculus. Notice that they have not been grouped to form formal tests: they are merely representative of items that may be used for diagnostic purposes at the start of the course or after a brief review of functions.

1. Describe the domain and range of each of the following functions and sketch their graphs. Also, note whether the functions are even, odd, periodic, or monotonic (in the whole domain). If periodic, give the fundamental (smallest) period.

   (a) $f : x \longrightarrow 7x - [3x] - [4x]$

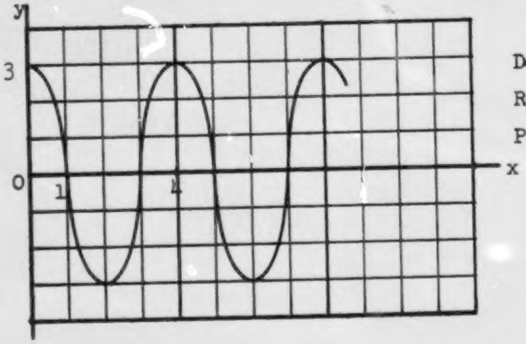


Domain: all reals.

Range: $\{y : 0 \leq y < 2\}$.

Period: 1 .

   (b) $f : x \longrightarrow 3 \sin \frac{\pi(x + 5)}{2}$



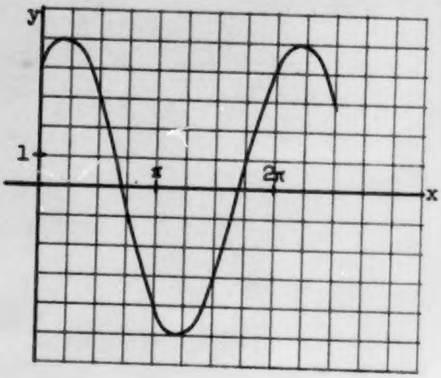
Domain: all reals.

Range: $[-3,3]$.

Period: 4 .

Even function.

(c) $f : x \longrightarrow 3 \sin x + 4 \cos x$



Domain: all reals.
Range: $[-5, 5]$.
Period: 2π .

To find maximum (range), let $\sin \alpha = \frac{4}{5}$ and then $\cos \alpha = \frac{3}{5}$.

$3 \sin x + 4 \cos x = 5[\cos \alpha \sin x + \sin \alpha \cos x] = 5 \sin(x + \alpha)$.

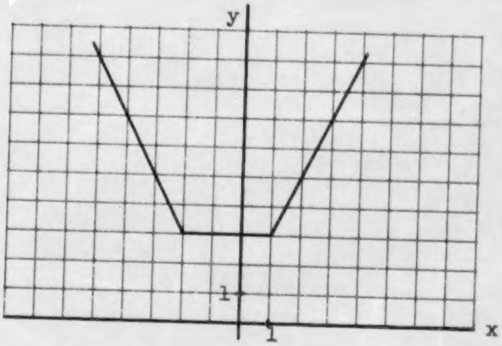
The maximum of the sin occurs at $\frac{\pi}{2}$.

If $x + \alpha = \frac{\pi}{2}$, $x = \frac{\pi}{2} - \arcsin \frac{4}{5} = \arccos \frac{4}{5} = \arcsin \frac{3}{5}$.

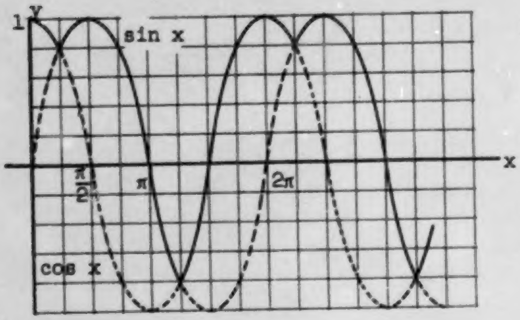
At this value of x , $f(x) = 3(\frac{3}{5}) + 4(\frac{4}{5}) = 5$.

(d) $f : x \longrightarrow |1 - x| + |2 + x|$

Domain: all reals.
Range: $y \geq 3$.



(e) $f : x \longrightarrow \max(\sin x , \cos x)$



Domain: all reals.

Range: $[1 , -\frac{\sqrt{2}}{2}]$.

Period: 2π .

Solid line is graph of $f : x \longrightarrow \max(\sin x , \cos x)$.

2. For each pair f and g of the following functions, find the composite functions fg and gf and state their domains and ranges (if the composites exist).

(a) $f : x \longrightarrow x^3$, $g : x \longrightarrow \sqrt{x + 8}$,

$fg : x \longrightarrow (x + 8)^{3/2}$; $gf : x \longrightarrow \sqrt{x^3 + 8}$;

Domain: $x \geq -8$;
Range: $y \geq 0$.

Domain $x \geq -2$;
Range: $y \geq 0$.

(b) $f : x \longrightarrow x - [x]$, $g : x \longrightarrow x^2$

$fg : x \longrightarrow x^2 - [x^2]$; $gf : x \longrightarrow (x - [x])^2$;

Domain: all reals;
Range: $0 \leq y < 1$.

Domain: all reals;
Range: $0 \leq y\ 1$.

(c) $f : x \longrightarrow \sin^2 x$, $g : x \longrightarrow \sqrt{x - 2}$

$fg : x \longrightarrow \sin^2 \sqrt{x - 2}$;

Domain: $x \geq 2$;
Range: $[0,1]$

gf does not exist;

Range of f is not included in domain of g .

(d) $f : x \longrightarrow \max(\sin \pi x\ ,\ \cos \pi x)$, $g : x \longrightarrow [x]$

$fg : x \longrightarrow \max(\sin \pi[x], \cos \pi\ [x])$;

Domain: all reals;

Range: $\{0,1\}$.

$gf : x \longrightarrow \left[\max(\sin \pi x\ ,\ \cos. \pi x)\right]$;

Domain: all reals;

Range: $\{-1,0,1\}$.

(e) $f : x \longrightarrow 1 + \operatorname{sgn} 2x$, $g : x \longrightarrow \cos \frac{3\pi x}{2}$

$fg : x \longrightarrow 1 + \operatorname{sgn} 2 \cos \frac{3\pi x}{2}$; $gf : x \longrightarrow \cos \frac{3\pi}{2}(1 + \operatorname{sgn} 2x)$;

Domain: all reals; Domain: all reals;

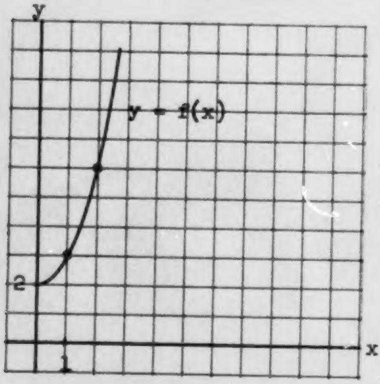
Range: $\{0,1,2\}$. Range: $\{-1,0,1\}$.

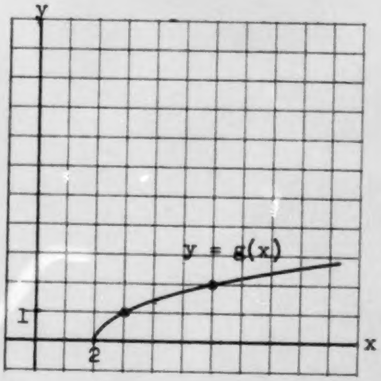
3. Which of the following functions when properly defined have inverses? Describe an inverse for each by means of a graph (sketch) and an equation. Also, give the domain and range of the function and its inverse.

(a) $f : x \longrightarrow x^2 + 2$

This function, when defined on a domain consisting of a subset of either of the two sets $(-\infty, 0]$ or $[0, \infty)$, will have an inverse. If domain of f is $[0, \infty)$, range is $[2, \infty)$, and inverse will be $g : x \longrightarrow \sqrt{x - 2}$.

Domain of $g = [2,\infty]$, and range $= [0,\infty)$.

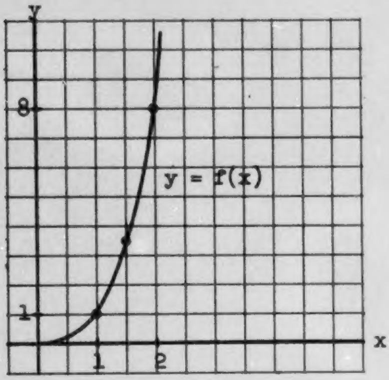


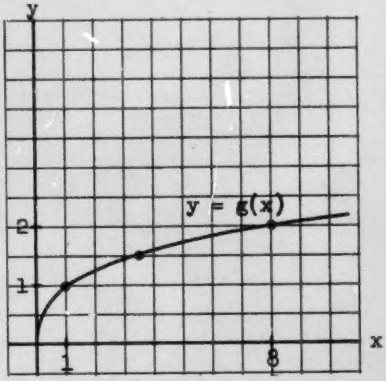


(b) $f : x \longrightarrow x^2|x|$

Eligible domains of f are subsets of either $(-\infty, 0]$ or $[0, \infty)$. If the domain of f is taken to be $[0,\infty)$, then its range is the same set.

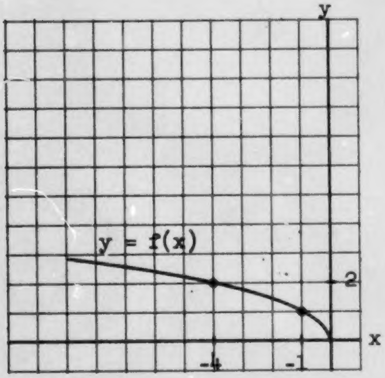
The inverse is $g : x \longrightarrow \sqrt[3]{x}$; domain = range = $[0,\infty)$.

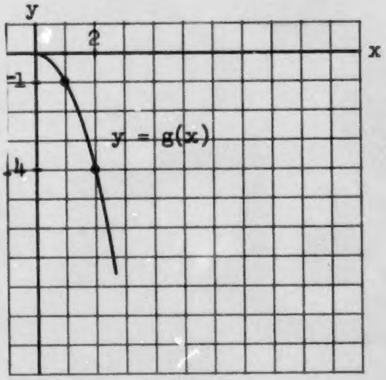




(c) $f : x \longrightarrow \sqrt{|x|}$.

Eligible domains of f are subsets of either $(-\infty,0]$ or $[0,\infty)$. If we choose the former, for example, then the range of f is $[0,\infty)$, and its inverse is $g : x \longrightarrow -x^2$. The domain of g is $[0,\infty)$ and its range is $(-\infty,0]$.





4. If $f(x) = (3x^5 + 1)^3$, find at least two different functions g such that $fg = gf$.

The function $g : x \longrightarrow x$ may be considered trivial, but it is valid. Another is $g : x \longrightarrow k$, where $k = (3k^5 + 1)^3$.

The inverse of f , defined by $g(x) = \sqrt[5]{\dfrac{\sqrt[3]{x} - 1}{3}}$, is also such a function.

5. Starting from the identities

(i) $\sin(x + y) = \sin x \cos y + \cos x \sin y$,

(ii) $\cos x = \sin(\frac{\pi}{2} - x)$,

(iii) $-\sin x = \sin(-x)$ and,

(iv) $\sin \frac{\pi}{2} = 1$, for all x and y ,

prove that

(a) $\cos(x + y) = \cos x \cos y - \sin x \sin y$.

From (iii) $-\sin 0 = \sin 0$, or $\sin 0 = 0$.

Then by (ii) , $0 = \sin 0 = \sin(\frac{\pi}{2} - \frac{\pi}{2}) = \cos \frac{\pi}{2} = 0$.

By (i) and (ii) , $\cos x = \sin\left(\frac{\pi}{2} + (-x)\right)$

$$= \sin \frac{\pi}{2} \cos(-x) + \cos \frac{\pi}{2} \sin(-x) ,$$

hence, $\cos x = \cos(-x)$.

Then by (ii) and (i), $\cos(x + y) = \cos(-x - y) = \sin(\frac{\pi}{2} + x + y)$

$$= \sin(\tfrac{\pi}{2} + x)\cos y + \cos(\tfrac{\pi}{2} + x)\sin y.$$

By (ii) $\sin(\frac{\pi}{2} + x) = \cos(-x) = \cos x$

and $\cos(\frac{\pi}{2} + x) = \sin(-x) = -\sin x$.

So $\cos(x + y) = \cos x \cos y - \sin x \sin y$.

(b) $\sin 3x = 3 \sin x \cos^2 x - \sin^3 x$

$\sin 3x = \sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x$.

But $\sin 2x = \sin(x + x) = 2 \sin x \cos x$,

$\cos 2x = \cos(x + x) = \cos^2 x - \sin^2 x$.

Hence, $\sin 3x = 2 \sin x \cos^2 x + \sin x \cos^2 x - \sin^3 x$

$= 3 \sin x \cos^2 x - \sin^3 x$.

(c) $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$

From the definition $\tan x = \dfrac{\sin x}{\cos x}$, we have

$$\tan 2x = \frac{\sin 2x}{\cos 2x} = \frac{2 \sin x \cos x}{\cos^2 x - \sin^2 x}$$

$$= \frac{2\,\dfrac{\sin x}{\cos x}}{1 - \dfrac{\sin^2 x}{\cos^2 x}} = \frac{2 \tan x}{1 - \tan^2 x} .$$

6. Evaluate:

(a) $\sin 5\pi = \sin \pi = 0$

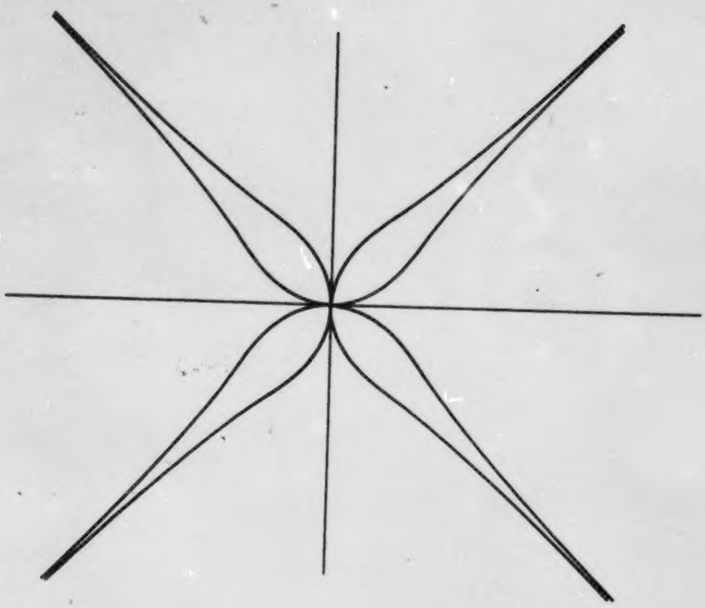
(b) $\cos \frac{\pi}{3} = \frac{1}{2}$

(c) $\tan \frac{5\pi}{3} = -\sqrt{3}$

(d) $\sin 31 \frac{\pi}{2} = \sin \frac{3\pi}{2} = -1$

(e) $\cos(-\frac{3\pi}{4}) = \cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$

7. Sketch, using polar coordinates, a graph of $r = \tan 2\theta$.



Teacher's Commentary

Appendix 3

MATHEMATICAL INDUCTION

TC A3-1. The Principle of Mathematical Induction.

The Principle of Mathematical Induction may be thought of as a postulate for the set of natural numbers N, rather than as a postulate about legitimate methods of proof (Metamathematics). Thus, we may state the principle in the following form:

Let M be a subset of N satisfying

(i) $1 \in M$,

(ii) if $n \in M$, then $n + 1 \in M$,

then $M = N$.

We can then deduce the form of the principle in the text by setting M equal to the set of natural numbers n for which A_n is true.

As stated here, the Principle of Mathematical Induction can be used to play a central role in the axiomatic development of the natural numbers. In Foundations of Analysis by E. Landau (Chelsea), the arithmetic of the natural, rational, real, and complex numbers is developed soley on the basis of the five postulates of Peano. The fifth postulate is the postulate of induction.

Solutions Exercises A3-1

The solutions of several of the exercises follow the same pattern for the sequential step. In each case after assuming A_k, we add an appropriate term to each side of the equation which is the expression of A_k, and show that the resulting equation reduces to A_{k+1}. For brevity we give only the solutions of two such exercises; these will be found below in 2 and 12.

1. Prove by mathematical induction that $1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n + 1)$.

Follow the pattern given in 2.

2. By mathematical induction prove the familiar result, giving the sum of an arithmetic progression to n terms:

$$a + (a + d) + (a + 2d) + \dots + \Big(a + (n - 1)d\Big) = \frac{n}{2}[2a + (n - 1)d] .$$

Initial Step. $a = \frac{1}{2}(2a + 0 \cdot d) = a$.

Sequential Step. Assume $A_k : a + (a + d) + (a + 2d) + \dots + [a + (k - 1)d]$ $= \frac{k}{2}[2a + (k - 1)d]$.

Add $a + kd$ to both sides getting

$$\begin{aligned} a + (a + d) + \dots + [a + (k - 1)d] + (a + kd) \\ &= \frac{k}{2}[2a + (k - 1)d] + (a + kd) \\ &= ka + \frac{k(k - 1)}{2} d + a + kd \\ &= (k + 1)a + \frac{k^2 - k + 2k}{2} d \\ &= \frac{(k + 1)}{2}(2a + kd) \\ &= \frac{k + 1}{2} [2a + [(k + 1) - 1]d] \end{aligned}$$

which is A_{k+1} .

This completes the proof.

3. By mathematical induction prove the familiar result, giving the sum of a geometric progression to n terms:

$$a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1} .$$

Follow the pattern given in 2 .

Prove the following four statements by mathematical induction.

4. $1^2 + 3^2 + 5^2 + \dots + (2n - 1)^2 = \frac{1}{3}(4n^3 - n)$.

Follow the pattern given in 2 .

5. $2n \leq 2^n$

Initial Step. $2 \cdot 1 \leq 2^1$.

Sequential Step. Assume $A_k : 2 \cdot k \leq 2^k$.

Then $2(k + 1) = 2k + 2 \leq 2k + 2k$, since $k \geq 1$.

$\leq 2 \cdot 2k \leq 2 \cdot 2^k$,

by the assumption A_k .

Therefore $2(k + 1) \leq 2^{k+1}$ which is A_{k+1} .

This completes the proof.

6. If $p > -1$; then, for every positive integer n , $(1 + p)^n \geq 1 + np$.

Initial Step. $(1 + p)^1 \geq 1 + 1 \cdot p$.

Sequential Step. Assume $A_k : (1 + p)^k \geq 1 + kp$.

Then, since $p > -1$, $1 + p > 0$, and we may multiply both sides of the inequality by $1 + p$ without changing its sense.

Therefore $(1 + p)^{k+1} \geq (1 + kp)(1 + p)$

$\geq 1 + kp + p + kp^2$ and dropping the positive quantity kp^2 we get $(1 + p)^{k+1} \geq 1 + (k + 1)p$ which is A_{k+1} .

This completes the proof.

7. $1 + 2 \cdot 2 + 3 \cdot 2^2 + \dots + n \cdot 2^{n-1} = 1 + (n - 1)2^n$.

Follow the pattern given in 2 .

Prove the following by the second principle of mathematical induction.

8. For all natural numbers n , the number $n + 1$ either is a prime or can be factored into primes.

We use the second principle of induction.

Initial Step. The number 2 is a prime.

Sequential Step. Let A_n be the statement of Exercise 8, and assume that A_s is true for all natural numbers s satisfying $s \leq k$. In other words, every integer $2, 3, 4, \ldots, k + 1$ is prime or a product of primes. In order to prove A_{k+1} we must show that $k + 2$ either is prime or can be factored into primes. If $k + 2$ is prime we are done. If not, we can write $k + 2$ as a product of factors r and t both less than $k + 2$, hence, both less than or equal to $k + 1$. By hypothesis, then, both factors r, t must be primes or products of primes. It follows that $k + 2$ can be written as a product of primes and that A_{k+1} is true.

9. For each natural number n greater than one, let U_n be a real number with the property that for at least one pair of natural numbers p, q with $p + q = n$, $U_n = U_p + U_q$.

When $n = 1$, we define $U_1 = a$ where a is some given real number. Prove that $U_n = na$ for all n.

Initial Step. $U_1 = 1 \cdot a$ by definition.

Sequential Step. Let A_n be the statement of Exercise 9. Using the second principle of induction we assume that for each number $s \leq k$ that A_s is true.
Now A_{k+1} must be established. But if U_{k+1} is a real number such that for at least one pair of natural numbers, f and g such that $f + g = k + 1$,

$$U_{k+1} = U_f + U_g ,$$

we know that f and g must each be less than or equal to k; and therefore U_f and U_g are real numbers to which the sequential hypothesis may be applied. Therefore

$$U_f = f \cdot a \quad \text{and} \quad U_g = g \cdot a$$

and so

$$U_{k+1} = f \cdot a + g \cdot a = (f + g) \cdot a = (k + 1)a$$

which is A_{k+1}.

This completes the proof.

10. Attempt to prove 8 and 9 from the first principle to see what difficulties arise.

In 8 note that the sequential step is based essentially upon the fact that r and t are each at most $k + 1$, not necessarily equal to $k + 1$. It would therefore be impossible to derive A_{k+1} from A_k alone and we cannot employ the first principle. Similarly in 9, we know only that f and g are at most k, not that they are necessarily equal to k. So we need to be able to refer to A_s for $s \leq k$, not to just A_k.

In the next three problems, first discover a formula for the sum, and then prove by mathematical induction that you are correct.

11. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n + 1)}$

To discover the formula for the sum, we might try writing down the sums in succession.

Thus

$$S_1 = \frac{1}{1 \cdot 2} = \frac{1}{2}$$

$$S_2 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

$$S_3 = \frac{2}{3} + \frac{1}{3 \cdot 4} = \frac{3}{4}$$

$$S_4 = \frac{3}{4} + \frac{1}{4 \cdot 5} = \frac{4}{5}$$

So we guess $S_n = \frac{n}{n + 1}$ and try to prove it.

For the proof follow the pattern of 2.

12. $1^3 + 2^3 + 3^3 + \ldots + n^3$. (Hint: Compare the sums you get here with Examples A3-1a and A3-1g in the text, or, alternatively, assume that the required result is a polynomial of degree 4.)

To guess the sum, we write down in succession the following:

$$S_1 = 1^3 = 1 = 1^2 = 1^2;$$

$$S_2 = 1 + 2^3 = 9 = 3^2 = (1 + 2)^2;$$

$$S_3 = 9 + 3^3 = 36 = 6^2 = (1 + 2 + 3)^2;$$

$$S_4 = 36 + 4^3 = 100 = 10^2 = (1 + 2 + 3 + 4)^2.$$

We guess therefore that $S_n = (1 + 2 + 3 + \dots + n)^2$. To prove this directly by induction is quite messy (try it), but if we remember from Number 1 that $1 + 2 + 3 + \dots + n = \frac{n(n + 1)}{2}$, we get a formula much easier to prove: $1^3 + 2^3 + \dots + n^3 = \frac{n^2(n + 1)^2}{4}$.

Now we follow the pattern of 2.

Initial Step. $1^3 = \frac{1^2 \cdot (1 + 1)^2}{4} = 1$.

Sequential Step. Assume $A_k : 1^3 + 2^3 + \dots + k^3 = \frac{k^2(k + 1)^2}{4}$. Add $(k + 1)^3$ to both sides, getting the following:

$$\begin{aligned} 1^3 + 2^3 + \dots + k^3 + (k + 1)^3 &= \frac{k^2(k + 1)^2}{4} + (k + 1)^3 \\ &= \frac{[(k + 1)^2(k^2 + 4(k + 1))]}{4} \\ &= \frac{(k + 1)^2(k + 2)^2}{4} \\ &= \frac{(k + 1)^2((k + 1) + 1)^2}{4}, \end{aligned}$$

which is A_{k+1}.

This completes the proof.

13. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n + 1)$. (Hint: Compare this with Example A3-1g in the text.)

To guess the sum we write down in succession the following:

$$\begin{aligned} S_1 &= 1 \cdot 2 &&= 2 \\ S_2 &= 2 + 2 \cdot 3 &&= 8 \\ S_3 &= 8 + 3 \cdot 4 &&= 20 \\ S_4 &= 20 + 4 \cdot 5 &&= 40 . \end{aligned}$$

This does not seem to be getting us very far. We try another approach. If you have worked Example A3-1g (and remember it) try writing S_n in this fashion,

$$\begin{aligned} S_n &= 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n + 1) \\ &= 1(1 + 1) + 2(2 + 1) + 3(3 + 1) + \dots + n(n + 1) \\ &= 1^2 + 1 + 2^2 + 2 + 3^2 + 3 + \dots + n^2 + n \\ &= (1^2 + 2^2 + 3^2 + \dots + n^2) + (1 + 2 + 3 + \dots + n) \\ &= \frac{n(n + 1)(2n + 1)}{6} + \frac{n(n + 1)}{2} = \frac{n(n + 1)(2n + 1 + 3)}{6} \\ &= \frac{n(n + 1)(n + 2)}{3} . \end{aligned}$$

Another way of guessing this formula would be to assume, as in Example A3-1g, that since the general term in S_n is quadratic, the formula might be cubic

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n + 1) = an^3 + bn^2 + cn + d$$

and then let n take on the successive values 1 , 2 , 3 , and 4 to determine a , b , c , and d . Thus, by successive subtractions,

$$\left.\begin{array}{l} a + b + c + d = 2 \\ 8a + 4b + 2c + d = 8 \\ 27a + 9b + 3c + d = 20 \\ 64a + 16b + 4c + d = 40 \end{array}\right\} \quad \left.\begin{array}{l} 7a + 3b + c = 6 \\ 19a + 5b + c = 12 \\ 37a + 7b + c = 20 \end{array}\right\} \quad \left.\begin{array}{l} 12a + 2b = 6 \\ 18a + 2b = 8 \end{array}\right\} \quad 6a = 2 .$$

Therefore

$$a = \frac{1}{3} , \ b = 1 , \ c = \frac{2}{3} , \ d = 0 ,$$

and

$$S_n = \frac{1}{3}n^3 + n^2 + \frac{2}{3}n = \frac{n(n + 1)(n + 2)}{3} .$$

The proof of these results follows the pattern of 2 and 12.

14. Prove for all positive integers n ,

$$\left(1 + \frac{3}{1}\right)\left(1 + \frac{5}{4}\right)\left(1 + \frac{7}{9}\right) \dots \left(1 + \frac{2n + 1}{n^2}\right) = (n + 1)^2 .$$

Check the initial step.

Assume A_k , and multiply both sides of the resulting equation by the appropriate factor, and reduce to get A_{k+1} .

15. Prove that $(1 + x)(1 + x^2)(1 + x^4) \ldots (1 + x^{2^n}) = \dfrac{1 - x^{2^{n+1}}}{1 - x}$.

Follow the pattern of Solution 14.

16. Prove that $n(n^2 + 5)$ is divisible by 6 for all integral n .

Initial Step. $1(1 + 5) = 6$ and this is divisible by 6 .
Assume $A_k : k(k^2 + 5) = 6p$ where p is a positive integer.
Consider:

$$\begin{aligned}(k + 1)\Big((k + 1)^2 + 5\Big) &= (k + 1)^3 + 5(k + 1)\\ &= k^3 + 3k^2 + 3k + 1 + 5k + 5\\ &= (k^3 + 5k) + (3k^2 + 3k) + 1 + 5\\ &= k(k^2 + 5) + 3k(k + 1) + 6 .\end{aligned}$$

By A_k we know that $k(k^2 + 5) = 6p$, and since k is a positive integer either k or $k + 1$ is an even integer. Therefore the second term is divisible both by 2 and by 3 , and therefore by 6 . Finally we get

$$\begin{aligned}(k + 1)\Big((k + 1)^2 + 5\Big) &= 6p + 6q + 6\\ &= 6(p + q + 1)\end{aligned}$$

and this finishes the proof, since we know that the sum of three positive integers is a positive integer.

17. Any infinite straight line separates the plane into two parts; two intersecting straight lines separate the plane into four parts; and three non-concurrent lines, of which no two are parallel, separate the plane into seven parts. Determine the number of parts into which the plane is separated by n straight lines of which no three meet in a single common point and no two are parallel; then prove your result. Can you obtain a more general result when parallelism is permitted? If concurrence is permitted? If both are permitted?

Both our method of guessing the answer and our proof will be sequential. Let R_n be the number of regions into which the plane is divided by n lines of which no two are parallel and no three are concurrent. If we draw an $(n + 1)$-th line under the same conditions, it must meet all the other lines in n new points of intersection. In crossing n lines it must go through $n + 1$ regions of the plane, dividing each region into two parts, thus adding $n + 1$ new regions. We conclude that

$$R_{n+1} = (n + 1) + R_n .$$

Since $R_1 = 2$, this is a recursive definition for R_n . We have, plainly,

$$R_n = 2 + 2 + 3 + 4 + \dots + n = \frac{1}{2}(n^2 + n + 2)$$

and this result can be obtained directly from the recursion formula by a straightforward induction.

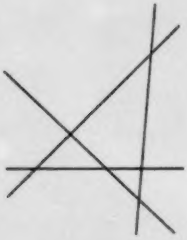
If parallelism is permitted, each pair of parallel lines existing reduces R_n by 1 , since one crossing is eliminated. Thus if p lines are parallel, you can pick $\frac{p(p - 1)}{2}$ pairs of parallel lines and there will be this many fewer regions

$$R_n = 1 + \frac{n(n + 1)}{2} - \frac{(p - 1)p}{2} .$$

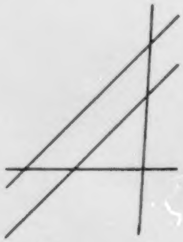
For example if four lines are drawn, three of which are parallel, there will be

$$1 + \frac{4(5)}{2} - \frac{3(2)}{2} = 8 \text{ regions.}$$

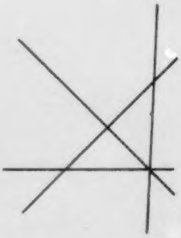
Similarly, any line which concurs with an already existing intersection point reduces the total number of intersection points by one, and the number of regions of the plane by one. Again we must remember, as in the parallel case, that pairs of extra concurrencies must all be counted.



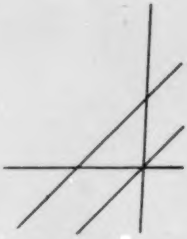
$n=4$, $p=0$, $c=0$
$R_4=11$



$n=4$, $p=2$, $c=0$
$R_4=10$



$n=4$, $p=0$, $c=3$
$R_4=10$



$n=4$, $p=2$, $c=3$
$R_4=9$

Thus if c lines concur at one point

$$R_n = 1 + \frac{n(n + 1)}{2} - \frac{(c - 1)(c - 2)}{2} .$$

If a line provides both a case of parallelism and a case of concurrence, it must be counted each way in reducing the number of regions, as is shown in the figure. In general if there are j families of parallel lines with $p_1, p_2, \dots, p_j$ lines in each family and k families of concurrent lines with $c_1, c_2, \dots, c_k$ lines in each family, we have

$$R_n = 1 + \frac{n(n+1)}{2} - \left[\frac{p_1(p_1-1)}{2} + \frac{p_2(p_2-1)}{2} + \dots + \frac{p_j(p_j-1)}{2}\right] - \left[\frac{(c_1-1)(c_1-2)}{2} + \frac{(c_2-1)(c_2-2)}{2} + \dots + \frac{(c_k-1)(c_k-2)}{2}\right]$$

The proof of this is too lengthy for insertion here.

18. Consider the sequence of fractions

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \dots, \frac{p_n}{q_n}, \dots$$

where each fraction is obtained from the preceding by the rule

$$p_n = p_{n-1} + 2q_{n-1}$$

$$q_n = p_{n-1} + q_{n-1} .$$

Show that for n sufficiently large, the difference between $\frac{p_n}{q_n}$ and $\sqrt{2}$ can be made as small as desired. Show also that the approximation to $\sqrt{2}$ is improved at each successive stage of the sequence and that the error alternates in sign. Prove also that p_n and q_n are relatively prime, that is, the fraction $\frac{p_n}{q_n}$ is in lowest terms.

Let the error at the n-th stage be denoted by $e_n = \frac{p_n}{q_n} - \sqrt{2}$. We may define the error e_{n+1} at the next stage recursively in terms of e_n as follows:

$$e_{n+1} = \frac{p_n + 2q_n}{p_n + q_n} - \sqrt{2}$$

$$= \frac{\frac{p_n}{q_n} + 2}{\frac{p_n}{q_n} + 1} - \sqrt{2}$$

$$= \frac{e_n + 2 + \sqrt{2}}{e_n + 1 + \sqrt{2}} - \sqrt{2}$$

$$= \frac{e_n(1 - \sqrt{2})}{e_n + 1 + \sqrt{2}} .$$

Since $1 - \sqrt{2}$ is negative, it follows that e_{n+1} has the opposite sign from e_n, and the sign alternates if the denominator is shown to be positive. We shall prove by induction that $|e_n| < \frac{1}{2^n}$ and thereby show simultaneously that the denominator above is positive, and that the error can be made as small as desired by taking n sufficiently large.

Initial Step. $|e_1| = |1 - \sqrt{2}| = .414 \ldots < \frac{1}{2}$.

Sequential Step. Assume $|e_k| < \frac{1}{2^n}$. For the denominator of e_{k+1}, we have

$$e_k + 1 + \sqrt{2} > -\frac{1}{2^k} + 1 + \sqrt{2} > -\frac{1}{2} + 1 + \sqrt{2} > \frac{1}{2} + \sqrt{2} > 1 .$$

We also have $$\sqrt{2} - 1 < \frac{1}{2} .$$

It follows from the recursive expression for e_{k+1} that

$$|e_{k+1}| < \frac{1}{2}|e_k| < \frac{1}{2} \cdot \frac{1}{2^k} = \frac{1}{2^{k+1}} .$$

To prove that p_n and q_n have no common factor other than 1, we note that

$$p_n = p_{n+1} - q_{n+1} , \quad q_n = 2q_{n+1} - p_{n+1} .$$

We then reason inductively as follows:

Initial Step. The only common factor of p_1 and of q_1 is 1 .

Sequential Step. Assume p_k and q_k have no common factor other than 1 . If p_{k+1} and q_{k+1} had such a common factor, then, by the above formula it would have to be a common factor of p_k and q_k . Contradiction.

19. Let p be any polynomial of degree m. Let $q(n)$ denote the sum

(1) $$q(n) = p(1) + p(2) + p(3) + \dots + p(n) .$$

Prove that there is a polynomial q of degree $m + 1$ satisfying (1).

Initial Step. We observe that if p has degree 0, then $p = c$ where c is a constant and we have

(1) $$p(1) + p(2) + \dots + p(n) = c + c + c + \dots + c = cn .$$

Hence $q(x) = cx$ is a polynomial of first degree satisfying the condition.

Sequential Step. We assume that the theorem is true for any polynomial p of degree less than or equal to k. Let

(2) $$p(x) = ax^{k+1} + p_1(x) , \qquad (a \neq 0)$$

where the degree of p_1 is $\leq k$.
Next we observe that

(3) $$(x + 1)^{k+2} = x^{k+2} + (k + 2)x^{k+1} + p_2(x)$$

where the degree of p_2 is $\leq k$. This fact has to be proved by induction, unless the binomial theorem is taken for granted. It will be proved afterward. Setting

$$(x + 1)^{k+2} - x^{k+2} = (k + 2)x^{k+1} + p_2(x)$$

and solving for x^{k+1} we obtain in (2)

(4) $$p(x) = \frac{a}{k + 2}[(x + 1)^{k+2} - x^{k+2}] + p_3(x)$$

where $$p_3(x) = p_1(x) - \frac{a}{k + 2} p_2(x)$$

and therefore the degree of $p_3(x)$ is $\leq k$.
Consequently,

$$p(1) + p(2) + \dots + p(n) = \frac{a}{k + 2} [(2^{k+2} - 1^{k+2}) + (3^{k+2} - 2^{k+2}) + \dots + [(n + 1)^{k+2} - n^{k+2}]] + p_3(1) + p_3(2) + \dots + p_3(n) .$$

By the induction hypothesis, there exists a $q_1(x)$ of degree $\leq k + 1$, such that

$$q_1(n) = p_3(1) + \dots + p_3(n) .$$

Furthermore the expression in braces reduces by successive additions and subtractions to $(n+1)^{k+2} - 1^{k+2}$, and we obtain the desired polynomial,

$$q(x) = \frac{a}{k+2}[(x+1)^{k+2} - 1^{k+2}] + q_1(x)$$

where $\quad q(n) = p(1) + \ldots + p(n)$.

Now we prove (3):

Initial Step. If $k = 0$, $(x+1)^2 = x^2 + (0+2)x + 1$ and the degree of 1 is 0 .

Sequential Step. $$\begin{aligned}(x+1)^{k+3} &= x^{k+2}(x+1) + (k+2)x^{k+1}(x+1) + (x+1)p_2(x) \\ &= x^{k+3} + (k+3)x^{k+2} + [(k+2)x^{k+1} + (x+1)p_2(x)] .\end{aligned}$$

20. Let the function $f(n)$ be defined recursively as follows:
Initial Step. $f(1) = 3$.
Sequential Step. $f(n+1) = 3^{f(n)}$
In particular, we have $f(3) = 3^{3^3} = 3^{27}$, etc.
Similarly, $g(n)$ is defined by
Initial Step. $g(1) = 9$.
Sequential Step. $g(n+1) = 9^{g(n)}$
Find the minimum value m for each n such that $f(m) \geq g(n)$.

It is easily seen that $g(n) > f(n)$ for all n . We shall prove that $f(n+1) > g(n)$ for all n and, hence, that $m = n+1$ is the least value for which $f(m) > g(n)$.

Initial Step. If $n = 1$, $f(2) = 3^3 = 27$, and $g(1) = 9$. Consequently $f(2) > g(1)$. More strongly, $f(2) > 2g(1) + 1$; and we shall prove generally $f(n+1) > 2g(n) + 1$.

Sequential Step. Suppose:

$$f(k+1) > 2g(k) + 1 > g(k) .$$

Then,

$$\begin{aligned} f(k+2) = 3^{f(k+1)} &> 3^{2g(k)+1} \geq 3 \cdot 3^{2g(k)} \\ &\geq 3 \cdot 9^{g(k)} \geq 3g(k+1) \\ &> 2g(k+1) + 1 \\ &> g(k+1) .\end{aligned}$$

21. Prove for all natural numbers n, that $\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$ is an integer. (Hint: Try to express $x^n - y^n$ in terms of $x^{n-1} - y^{n-1}$, $x^{n-2} - y^{n-2}$, etc.)

Let $F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$. We will use the second principle.

Initial Step.

$$F_1 = \frac{(1+\sqrt{5}) - (1-\sqrt{5})}{2\sqrt{5}} = \frac{1+\sqrt{5}-1+\sqrt{5}}{2\sqrt{5}} = 1$$

$$F_2 = \frac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{2^2\sqrt{5}} = \frac{1+2\sqrt{5}+5-1+2\sqrt{5}-5}{4\sqrt{5}}$$

$$= \frac{4\sqrt{5}}{4\sqrt{5}} = 1 .$$

Sequential Step. Assume F_s is an integer for all $s \leq k$.

Consider $F_{k+1} = \frac{(1+\sqrt{5})^{k+1} - (1-\sqrt{5})^{k+1}}{2^{k+1}\sqrt{5}}$. For brevity we write

$$1+\sqrt{5} = x \quad \text{and} \quad 1-\sqrt{5} = y .$$

Then

$$F_{k+1} = \frac{x^{k+1} - y^{k+1}}{2^{k+1}\sqrt{5}} = \frac{x^{k+1} + x^k y - x^k y + xy^k - xy^k - y^{k+1}}{2^{k+1}\sqrt{5}}$$

$$= \frac{x^k(x+y) - xy(x^{k-1} - y^{k-1}) - y^k(x+y)}{2^{k+1}\sqrt{5}}$$

$$= \frac{(x+y)(x^k - y^k)}{2^{k+1}\sqrt{5}} - \frac{xy(x^{k-1} - y^{k-1})}{2^{k+1}\sqrt{5}}$$

$$= \frac{x+y}{2}\left(\frac{x^k - y^k}{2^k\sqrt{5}}\right) - \frac{xy}{4}\left(\frac{x^{k-1} - y^{k-1}}{2^{k-1}\sqrt{5}}\right)$$

$$= \frac{(1+\sqrt{5}) + (1-\sqrt{5})}{2}F_k - \frac{(1+\sqrt{5})(1-\sqrt{5})}{4}F_{k-1}$$

$$= F_k + F_{k-1} .$$

but by the assumption of the sequential step we know F_k and F_{k-1} are integers. Therefore F_{k+1} is an integer. This complētes the theorem.

Solutions Exercises A3-2a

1. Prove

$$\sum_{k=1}^{n} (\alpha f_k + \beta g_k) = \alpha \sum_{k=1}^{n} f_k + \beta \sum_{k=1}^{n} g_k$$

The linearity of summation is a consequence of the additive and multiplicative properties of real numbers and follows easily by mathematical induction.

2. Write each of the following sums in expanded form and evaluate:

(a) $\sum_{k=1}^{5} 2k \qquad 2 + 4 + 6 + 8 + 10 = 30$

(b) $\sum_{j=2}^{6} j^2 \qquad 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 90$

(c) $\sum_{r=-1}^{3} (r^2 + r - 12) \qquad (-12) + (-12) + (-10) + (-6) + (0) = -40$

(d) $\sum_{m=2}^{5} m(m-1)(m-2) \qquad 0 + 3 \cdot 2 \cdot 1 + 4 \cdot 3 \cdot 2 + 5 \cdot 4 \cdot 3 = 90$

(e) $\sum_{i=0}^{10} 2^i \qquad 1 + 2 + 2^2 + 2^3 + \dots + 2^{10} = 2^{11} - 1 = 2043$

(f) $\sum_{r=0}^{4} \frac{4!}{r!(4-r)!} \qquad 1 + 4 + 6 + 4 + 1 = 16$

3. Which of the following statements are true and which are false? Justify your conclusions.

(a) $\sum_{j=3}^{10} 4 = 7 \cdot 4 = 28$

(b) $\sum_{j=m}^{n} 4 = 4(n - m + 1)$

(c) $\sum_{k=1}^{10} k^2 = 10 \sum_{k=1}^{9} k^2$

(d) $\sum_{k=1}^{1000} k^2 = 5 + \sum_{k=3}^{1000} k^2$

(e) $\sum_{k=1}^{n} k^3 = n^3 + \sum_{j=2}^{n} (j - 1)^3$

(f) $\sum_{m=1}^{10} k^2 = \left(\sum_{m=1}^{10} k \right)^2$

(g) $\sum_{m=1}^{10} k^3 = \left(\sum_{m=1}^{10} k \right)^2$

(h) $\sum_{i=0}^{n} i(i - 1)(n - i) = \sum_{i=2}^{n-1} i(i - 1)(n - i)$

(i) $\sum_{k=0}^{m} f(a_{m-k}) = \sum_{k=0}^{m} f(a_k)$

(j) $n \sum_{k=0}^{n} A_k - \sum_{k=0}^{n} kA_k = \sum_{k=0}^{n} kA_{n-k}$

$\wedge$ (k) $\sum_{k=0}^{m} k^2(A_k - A_{m-k}) = m^2 \sum_{k=0}^{m} A_{m-k} - 2m \sum_{k=0}^{m} k\, A_{m-k}$

(a) False; $\sum_{j=3}^{10} 4 = 8 \cdot 4 = 32$

(b) True

(c) False; $\sum_{k=1}^{10} k^2 = 10^2 + \sum_{k=1}^{9} k^2 < 10 \sum_{k=1}^{9} k^2$

(d) True

(e) True

(f) False; unless $k = 0$.

(g) False; unless $k = 0$.

(h) True; the missing terms are zero.

(i) True; $m - k$ takes on the same values as k .

(j) True: $\sum_{k=0}^{n} k\, A_{n-k} = \sum_{k=0}^{n} (n - k)A_k$ by (i).

(k) True; follows by applying (i) to $\sum_{k=0}^{m} (m^2 - 2mk + k^2)A_{m-k}$.

4. Evaluate $\sum_{k=1}^{n} f(\frac{k}{n})(\frac{b - a}{n})$ if $f(x) = x^2$, $a = 0$, $b = 1$ and

(a) $n = 2$. $\frac{5}{8}$

(b) $n = 4$. $\frac{15}{32}$

(c) $n = 8$. $\frac{102}{256}$

5. Subdivide the interval $[0,1]$ into n equal parts. In each subinterval obtain upper and lower bounds for x^2. Using sigma notation, use these upper and lower bounds to obtain expressions for upper and lower estimates of the area under the curve $y = x^2$ on $[0,1]$. If you can evaluate these sums without reading elsewhere, do so.

$$\text{Lower sum} = \sum_{k=0}^{n-1} \frac{1}{n}\left(\frac{k}{n}\right)^2 = \frac{1}{n^3}\sum_{k=0}^{n-1} k^2 = \frac{(n-1)(n)(2n-1)}{6n^3} .$$

$$\text{Upper sum} = \sum_{k=1}^{n} \frac{1}{n}\left(\frac{k}{n}\right)^2 = \text{lower sum} + \frac{1}{n} .$$

6. (a) Write out the sum of the first 7 terms of an arithmetic progression with first term a and common difference d. Express the same sum in sigma notation.

$$(a) + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) + (a + 5d) + (a + 6d)$$

$$= \sum_{n=1}^{7} [a + (n - 1)d] .$$

(b) In sigma notation, write the expression for the sum of the first n terms of a geometric progression with first term a and common ratio r.

$$\sum_{k=1}^{n} ar^{k-1}$$

7. (a) Consider a function f defined by

$$f(n) = \sum_{r=1}^{n} [(r - 1)(r - 2)(r - 3)(r - 4)(r - 5) + r] .$$

Find $f(n)$ for $n = 1, 2, \ldots, 5$.

$$f(n) = \frac{n(n + 1)}{2} \quad \text{for } n = 1, 2, \ldots, 5 .$$

(b) Give an example of a function g (similar to that in (a)) such that

$$g(n) = 1 \ , \ n = 1 \ , \ 2 \ , \ \ldots \ , \ 10^6 \ ,$$
$$g(10^6 + 1) = 0 \ .$$

$$g(n) = 1 - \frac{(n-1)(n-2)(n-3)\ \ldots\ (n-10^6)}{10^6!}$$

8. Write each of the following sums in expanded form and evaluate:

(a) $\displaystyle\sum_{n=1}^{4} \left\{ \sum_{r=1}^{3} r(n-r) \right\}$.

$$\sum_{n=1}^{4} \{1(n-1) + 2(n-2) + 3(n-3)\} = \sum_{n=1}^{4} \{6n - 14\} = 4$$

(b) $\displaystyle\sum_{n=1}^{N} \left\{ \sum_{r=1}^{R} (rn - 1) \right\}$

$$\sum_{n=1}^{N} \{(n-1) + (2n-1) + \ldots + (Rn-1)\} = \sum_{n=1}^{N} \frac{n(R)(R+1)}{2} - R$$

$$= \frac{N(N+1)(R)(R+1)}{4} - RN$$

∧9. The double sum $\displaystyle\sum_{i=0}^{m} \sum_{j=0}^{n} F(i,j)$ is a shorthand notation for

$$\sum_{i=0}^{m} \{F(i,0) + F(i,1) + \ldots + F(I,n)\} \quad \text{or} \quad \begin{array}{l} F(0,0) + F(0,1) + \ldots + F(0,n) \\ + F(1,0) + F(1,1) + \ldots + F(1,n) \\ \quad\ \vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \\ + F(m,0) + F(m,1) + \ldots + F(m,n) \ . \end{array}$$

In particular,

A3-2 a

$$\sum_{i=1}^{2}\sum_{j=1}^{3} i\cdot j = 1\cdot 1 + 1\cdot 2 + 1\cdot 3 + 2\cdot 1 + 2\cdot 2 + 2\cdot 3 = 18$$

Evaluate:

(a) $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} i\cdot j$ (c) $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} \max\{i,j\}$

(b) $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} (i+j)$ (d) $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} \min\{i,j\}$

(a) $$\sum_{i=1}^{m} i\left\{\sum_{j=1}^{n} j\right\} = \frac{m(m+1)(n)(n+1)}{4}$$

(b) $$\sum_{i=1}^{m}\left(ni + \frac{n(n+1)}{2}\right) = \frac{n(m)(m+1)}{2} + \frac{mn(n+1)}{2} = \frac{mn(m+n+2)}{2}\,.$$

(c) If $n \geq m$, we have

$$\sum_{i=1}^{m}\left\{\sum_{j=1}^{i} i + \sum_{j=i+1}^{n} j\right\} = \sum_{i=1}^{m}\left(i^2 + \frac{n(n+1)}{2} - \frac{i(i+1)}{2}\right)$$

$$= \frac{1}{2}\sum_{i=1}^{m} i(i-1) + \frac{1}{2}\sum_{i=1}^{m} n(n+1)$$

$$= \frac{(m-1)m(m+1)}{6} + \frac{mn(n+1)}{2}$$

For $m \geq n$, just interchange m and n (by symmetry).

(d) This can also be done similarly, i.e.,

$$\sum_{i=1}^{m}\left\{\sum_{j=1}^{i} j + \sum_{j=i+1}^{n} i\right\}$$

Alternatively,

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\max(i,j) + \min(i,j)\right) = \sum_{i=1}^{m}\sum_{j=1}^{n}(i+j)$$

Now use (b) and (c) , to give

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \min(i,j) = \frac{m^2 n}{2} - \frac{(m-1)(m)(m+1)}{6} .$$

10. (a) Show that $\frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$, $k \neq 0 , 1$.

$$\frac{1}{k-1} - \frac{1}{k} = \frac{k-(k-1)}{(k-1)k} = \frac{1}{k(k-1)} , \; k \neq 0 .$$

(b) Evaluate $\sum_{k=2}^{1000} \frac{1}{k(k-1)}$.

$$\sum_{k=2}^{1000} \frac{1}{k(k-1)} = \sum_{k=2}^{1000} \left(\frac{1}{k-1} - \frac{1}{k}\right) = \frac{1}{2-1} - \frac{1}{1000} = \frac{999}{1000} .$$

In general, $\sum_{k=2}^{n} \frac{1}{k(k-1)} = 1 - \frac{1}{n}$.

11. If $S(n) = \sum_{i=1}^{n} f(i)$, determine $f(m)$ in terms of the sum function S .

$f(1) = S(1)$

$S(m) - S(m-1) = \sum_{i=1}^{m} f(i) - \sum_{i=1}^{m-1} f(i) = f(m)$, $m > 1$.

12. Determine $f(m)$ in the following summation formulae: (See Number 11.)

(a) $1 = \sum_{i=1}^{n} f(i)$ $\qquad$ $f(1) = 1$, $f(m) = 0$, $m > 1$.

(b) $n = \sum_{i=1}^{n} f(i)$

$f(m) = 1 \ , \ m \geq 1 \ .$

(c) $n^2 = \sum_{i=1}^{n} f(i)$

$f(m) = m^2 - (m - 1)^2 = 2m - 1 \ , \ m \geq 1 \ .$

(d) $an^2 + bn + c = \sum_{i=1}^{n} f(i)$

$$\begin{aligned} f(1) &= a + b + c \ , \\ f(m) &= am^2 + bm + c - a(m-1)^2 - b(m-1) - c \\ &= a(2m - 1) + b \ , \ m > 1 \ . \end{aligned}$$

(e) $\cos nx = \sum_{i=1}^{n} f(i)$

$$\begin{aligned} f(1) &= \cos x \ , \\ f(m) &= \cos mx - \cos(m - 1)x \\ &= -2 \sin \frac{x}{2} \sin(m - \frac{1}{2})x \ , \ m > 1 \ . \end{aligned}$$

(f) $\sin(an + b) = \sum_{i=1}^{n} f(i)$

$$\begin{aligned} f(1) &= \sin(a + b) \ , \\ f(m) &= \sin(am + b) - \sin\left(a(m - 1) + b\right) \\ &= 2 \sin \frac{a}{2} \cos\left(am + b - \frac{a}{2}\right) \ , \ m > 1 \ . \end{aligned}$$

(g) $n! = \sum_{i=1}^{n} f(i)$

$$\begin{aligned} f(1) &= 1 \ , \\ f(m) &= m! - (m - 1)! \\ &= (m - 1)!(m - 1) \ , \ m > 1 \ . \end{aligned}$$

13. Binomial Theorem:

We define $\binom{n}{r} = \frac{n!}{(n - r)!r!}$ where r , n are integers such that $0 \leq r \leq n$. Also $0! = 1$ and $\binom{n}{r} = 0$ if $r > n$. Show that

(a) $\binom{n}{0} = \binom{n}{n} = 1$

$\binom{n}{1} = \binom{n}{n-1} = n$

$$\binom{n}{0} = \frac{n!}{n!0!} = \frac{n!}{n! \cdot 1} = 1$$

$$\binom{n}{n} = \frac{n!}{0!n!} = 1$$

$$\binom{n}{1} = \frac{n!}{(n-1)!1!} = \frac{n!}{(n-1)!(n-(n-1))!}$$

$$= \frac{n!}{(n-1)!} = n$$

(b) $\binom{n}{r} = \binom{n}{n-r}$

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n!}{(n-r)!(n-(n-r))!}$$

$$= \binom{n}{n-r}$$

$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$

$$\binom{n}{r} + \binom{n}{r+1}$$

$$= \frac{n!}{(n-r)!r!} + \frac{n!}{(n-r-1)!(r+1)!}$$

$$= \frac{n!(r+1)}{(n-r)!(r+1)!} + \frac{n!(n-r)}{(n-r)!(r+1)!}$$

$$= \frac{n!(n+1)}{(n-r)!(r+1)!} = \frac{(n+1)!}{(n-r)!(r+1)!}$$

$$= \binom{n+1}{r+1}$$

Λ(c) Establish the Binomial Theorem

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r = x^n + nx^{n-1} y + \ldots + nxy^{n-1} + y^n, \quad n = 0, 1, 2, \ldots$$

by mathematical induction.

For $n = 1$; $(x + y) = \binom{1}{0} xy^0 + \binom{1}{1} x^0 y = x + y$.

Now assume $(x + y)^k = \sum_{r=0}^{k} \binom{k}{r} x^{k-r} y^r$ for all $k \leq n$.

We show this implies the truth of the theorem for $k = n + 1$.

$$(x + y)^{n+1} = (x + y)^n(x + y) = \left(\sum_{r=0}^{n} \binom{n}{r} x^{n-r}\ y^r\right)(x + y)$$

$$= \sum_{r=0}^{n}\left(\binom{n}{r} x^{n-r+1}\ y^r + \binom{n}{r} x^{n-r}\ y^{r+1}\right)$$

$$= \sum_{r=1}^{n}\left(\left(\binom{n}{r} + \binom{n}{r-1}\right) x^{n-r+1}\ y^r\right) + \binom{n}{0} x^{n+1}\ y^0 + \binom{n}{n} x^0\ y^{n+1}$$

(using (b))

$$= \sum_{r=1}^{n} \binom{n+1}{r} x^{n-r+1}\ y^r + \binom{n+1}{0} x^{n+1}\ y^0 + \binom{n+1}{n+1} x^0\ y^{n+1}$$

$$= \sum_{r=0}^{n+1} \binom{n+1}{r} x^{n+1-r}\ y^r$$

$$= \sum_{r=0}^{k} \binom{k}{r} x^{k-r}\ y^k \text{ , where } k = n + 1 .$$

14. Using the binomial theorem, give the expansions for the following:

(a) $(x + y)^3$

$$(x + y)^3 = \binom{3}{3}x^3 + \binom{3}{2}x^2y + \binom{3}{1}xy^2 + \binom{3}{0}y^3$$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

(b) $(x - y)^3$

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - 3y^3$$

(c) $(2x - 3y)^3$

$$(2x - 3y)^3 = \binom{3}{3}(2x)^3 + \binom{3}{2}(2x)^2(-3y)$$

$$+ \binom{3}{1}(2x)(-3y)^2 + \binom{3}{0}(-3y)^3$$

$$= 8x^3 - 36x^2y + 54xy^2 - 23y^3$$

(d) $(x - 2y)^5$

$$(x - 2y)^5 = \binom{5}{5}x^5 + \binom{5}{4}x^4(-2y) + \binom{5}{3}x^3(-2y)^2$$

$$+ \binom{5}{2}x^2(-2y)^3 + \binom{5}{1}x(-2y)^4 + \binom{5}{0}(-2y)^5$$

$$= x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5$$

15. Evaluate the following sums.

(a) $\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = \sum_{r=0}^{n} \binom{n}{r}$

Since $(1 + x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$, $2^n = \sum_{r=0}^{n} \binom{n}{r}$ by setting $x = 1$.

(b) $\binom{n}{0} - \binom{n}{1} + \dots + (-1)^n \binom{n}{n} = \sum_{r=0}^{n} (-1)^r \binom{n}{r}$

If instead, we let $x = -1$, $0 = \sum_{r=0}^{n} (-1)^r \binom{n}{r}$.

16. Sum $\sum_{r=0}^{n} r\binom{n}{r}$ by first showing $\sum_{r=0}^{n} r\binom{n}{r} = \sum_{r=0}^{n} (n - r)\binom{n}{r}$ and using 15(a).

By 13(a), $\sum_{r=0}^{n} r\binom{n}{r} = \sum_{r=0}^{n} (n - r)\binom{n}{n-r} = \sum_{r=0}^{n} (n - r)\binom{n}{r}$.

Thus, $2\sum_{r=0}^{n} r\binom{n}{r} = n\sum_{r=0}^{n} \binom{n}{r} = n \cdot 2^n$ by 15(a), and the sum is $n \cdot 2^{n-1}$.

Λ17. If $P_n(x)$ denotes a polynomial of degree n such that $P_n(x) = 2^x$ for $x = 0 , 1 , 2 , \dots , n$ find $P_n(n + 1)$.

For this problem, it will be convenient to set

$$Q_r(x) = \frac{x(x - 1) \dots (x - r + 1)}{r!}$$

where r is a non-negative integer. Note for any integer $n \geq r$ that $Q_r(n) = \binom{n}{r}$. Consider the n-th degree polynomial

$$P_n(x) = \sum_{r=0}^{n} Q_r(x) = 1 + x + \frac{x(x-1)}{1 \cdot 2} + \dots + \frac{x(x-1)\dots(x-r+1)}{r!}$$

$P_n(0) = \binom{0}{0} = 1$,

$P_n(1) = \binom{1}{0} + \binom{1}{1} = 2$,

$P_n(2) = \binom{2}{0} + \binom{2}{1} + \binom{2}{2} + 2^2$,

$P_n(x) = 2^x$ for $x = 0, 1, 2, \dots, n$ by Number 15(a), and thus satisfies our requirements.

$$\begin{aligned} P_n(n+1) &= \binom{n+1}{0} + \binom{n+1}{1} + \dots + \binom{n+1}{n} \\ &= \sum_{r=0}^{n+1} \binom{n+1}{r} - \binom{n+1}{n+1} \\ &= 2^{n+1} - 1 . \end{aligned}$$

Solutions Exercises A3-2b

1. Write the following sums in telescoping form, i.e., in the form $\sum_{k=1}^{n} \{u(k) - u(k-1)\}$, and evaluate

(a) $\sum_{k=1}^{n} k(k+1)$

(b) $\sum_{k=1}^{n} k(2k-1)$

(c) $\sum_{k=1}^{n} 2k(2k+1)$

(d) $\sum_{k=1}^{n} k(k+1)(k+2)$

(e) $\sum_{k=1}^{n} k^3$

(f) $\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)}$

(g) $\sum_{k=1}^{n} k \cdot k!$

(h) $\sum_{k=1}^{n} r^k$

(a) $\frac{1}{3}\sum_{k=1}^{n}\{k(k+1)(k+2) - (k-1)k(k+1)\} = \frac{n(n+1)(n+2)}{3}$

(b) $k(2k-1) = 2k(k+1) - 3k$. Using (a) and $3k = \frac{3}{2}\{k(k+1) - (k-1)k\}$, the sum is $\frac{2n(n+1)(n+2)}{3} - \frac{3n(n+1)}{2}$.

(c) $2k(2k+1) = 4k(k+1) - 2k$. Using (a) and (b) , the sum is $\frac{4n(n+1)(n+2)}{3} - n(n+1)$.

(d) Here, $u(k) = \frac{k(k+1)(k+2)(k+3)}{4}$ and the sum is $\frac{n(n+1)(n+2)(n+3)}{4}$.

(e) $k^3 = k(k+1)(k+2) - 3k(k+1) - k$. Whence
$u(k) = \frac{k(k+1)(k+2)(k+3)}{4} - \frac{3k(k+1)(k+2)}{3} - \frac{k(k+1)}{2}$
and the sum is
$\frac{n(n+1)(n+2)(n+3)}{4} - n(n+1)(n+2) - \frac{n(n+1)}{2}$.

(f) Here, $u(k) = -\frac{1}{2(k+1)(k+2)}$ and the sum is $\frac{1}{2}\{\frac{1}{1\cdot 2} - \frac{1}{(n+1)(n+2)}\}$.

(g) Here, $u(k) = (k+1)!$ and the sum is $(n+1)! - 1$.

(h) Here, $u(k) = \frac{r^{k+1}}{r-1}$ and the sum is $\frac{r^{n+1} - 1}{r-1}$, $r \neq 1$.

2. Using $\sum_{k=1}^{n}\{u(k) - u(k-1)\} = u(n) - u(0)$, establish a short dictionary of summation formulae by considering the following functions u :
(a) $(a + kd)(a + (k+1)d) \ldots (a + (k+p)d)$
(b) The reciprocal of (a) .
(c) r^k
(d) kr^k
(e) k^2r^k
(f) $k!$
(g) $(k!)^2$
(h) $\arctan k$
(i) $k \sin k$

(a) $$(p + 1)d\sum_{k=1}^{n}(a + kd)(a + (k + 1)d)\ \dots\ (a + (k + p - 1)d)$$

$$= (a + nd)(a + (n + 1)d)\dots(a + (n + p)d) - a(a + d)\dots(a + pd) .$$

(b) $$(p + 1)d\sum_{k=1}^{n}\{(a + (k - 1)d)(a + kd)\dots(a + (k + p)d)\}^{-1}$$

$$= \{a(a + d)\dots(a + pd)\}^{-1} - \{(a + nd)(a + (n + 1)d)\dots(a + (k + p)d)\}^{-1}.$$

(c) $$(r - 1)\sum_{k=1}^{n} r^{k-1} = r^{n} - 1 .$$

(d) $$(r - 1)\sum_{k=1}^{n} kr^{k-1} + \sum_{k=1}^{n} r^{k-1} = nr^{n} \quad \text{or}$$

$$\sum_{k=1}^{n} kr^{k-1} = \frac{n(r - 1)r^{n} - r^{n} + 1}{(r - 1)^{2}} .$$

(e) $$(r - 1)\sum_{k=1}^{n} k^{2}r^{k-1} = n^{2}r^{n} + \sum_{k=1}^{n} r^{k-1} - 2\sum kr^{k-1} .$$

(Now use (c) and (d))

(f) $$\sum_{k=1}^{n}(k - 1)!(k - 1) = n! - 1 .$$

(g) $$\sum_{k=1}^{n}(k^{2} - 1)\{(k - 1)!\}^{2} = (n!)^{2} - 1$$

(h) $$\sum_{k=1}^{n}\arctan\frac{1}{k^{2} - k + 1} = \arctan\ n$$

(i) $$\tfrac{1}{2}(\sin\tfrac{1}{2})\sum_{k=1}^{n} k\cos(k - \tfrac{1}{2}) = n\sin n - \sum_{k=1}^{n}\sin(k - 1)$$

(Now use Equation 8)

3. Simplify:

$$\frac{\sin x + \sin 3x + \dots + \sin((2n - 1)x)}{\cos x + \cos 3x + \dots + \cos((2n - 1)x)} .$$

Since

$$\sum_{k=1}^{n} \cos(ak + b - \tfrac{a}{2}) = \cos(b + \tfrac{an}{2}) \frac{\sin \frac{an}{2}}{\sin \frac{a}{2}} ,$$

$$\sum_{k=1}^{n} \sin x(2k - 1) = \frac{\sin xn \sin xn}{\sin x}$$

by letting $a = -2x$, $b = \frac{\pi}{2}$, and

$$\sum_{k=1}^{n} \cos x(2k - 1) = \frac{\cos xn \sin xn}{\sin x}$$

by letting $a = 2x$, $b = 0$.

Whence,

$$\frac{\sum_{k=1}^{n} \sin x(2k - 1)}{\sum_{k=1}^{n} \cos x(2k - 1)} = \tan xn .$$

4. Another method for summing $\sum P(k)$ (P - a polynomial) can be obtained by using a special case of Number 2a, i.e.,

$$\sum_{k=1}^{n} \{(k + 1)(k)(k - 1)\dots(k - r + 1) - (k)(k - 1)(k - 2)\dots(k - r)\}$$
$$= (n + 1)(n)(n - 1)\dots(n - r + 1) ,$$

or

$$\sum_{k=1}^{n} k(k - 1)\dots(k - r + 1) = \frac{(n + 1)(n)(n - 1)\dots(n - r + 1)}{r + 1}.$$

First, we show how to represent any polynomial $P(k)$ of r-th degree in the form

$$(1)\quad P(k) = a_0 + a_1 k + \frac{a_2 k(k - 1)}{2!} + \dots + \frac{a_r k(k - 1)\dots(k - r + 1)}{r!}.$$

If $k = 0$, then $a_0 = P(0)$; if $k = 1$, then $a_1 = P(1) - P(0)$; if $k = 2$, then $a_2 = P(2) - 2\,P(1) + P(0)$. In general, it can be shown that

(ii) $a_m = P(m) - \binom{m}{1}P(m - 1) + \binom{m}{2}P(m - 2) - \dots + (-1)^m P(0)$,

$m = 0 , 1 , \dots , r$.

Since both sides of (i) are polynomials of degree r and (i) is satisfied for $m = 0 , 1 , \dots , r$, it must be an identity.

Now sum $\sum_{k=1}^{n} P(k)$.

$$\sum_{k=1}^{n} P(k) = a_0 n + \frac{a_1(n + 1)(n)}{2!} + \dots + \frac{a_r(n + 1)(n)(n - 1)\dots(n - r + 1)}{r!} .$$

5. Using Problem 4, find the following sums:

(a) $\sum_{k=1}^{n} k^2$

$k^2 = a_0 + a_1 k + \frac{a_2 k(k - 1)}{2!}$ where $a_0 = 0^2$, $a_1 = 1^2 - 0^2 = 1$, $a_2 = 2^2 - 2(1) + 0 = 2$. Thus, $k^2 = k + k(k - 1)$ and

$$\sum_{k=1}^{n} k^2 = \frac{n(n + 1)}{2} + \frac{(n + 1)(n)(n - 1)}{3} = \frac{n(n + 1)(2n + 1)}{6} .$$

(b) $\sum_{k=1}^{n} k^3 - \left(\sum_{k=1}^{n} k\right)^2$

$k^3 = a_0 + a_1 k + \frac{a_2 k(k - 1)}{2!} + \frac{a_3 k(k - 1)(k - 2)}{3!}$ where $a_0 = 0$, $a_1 = 1$, $a_2 = 2^3 - 2(1) = 6$, $a^3 = 3^3 - 3(8) + 3(1) = 6$. Thus, $k^3 = k + 3k(k - 1) + k(k - 1)(k - 2)$ (compare with Number 1e) and

$$\sum_{k=1}^{n} k^3 = \frac{(n + 1)n}{2} + \frac{3(n + 1)(n)(n - 1)}{3} + \frac{(n + 1)(n)(n - 1)(n - 2)}{4}$$

$$= \frac{n^2(n + 1)^2}{4} .$$

$$\sum_{k=1}^{n} k = \frac{(n + 1)(n)}{2} . \text{ Finally, } \sum_{k=1}^{n} k^3 - \left(\sum_{k=1}^{n} k\right)^2 = 0 .$$

(c) $\displaystyle\sum_{k=1}^{n} k^4$

$$k^4 = a_0 + a_1 k + \dots + \frac{a_4 k(k - 1)(k - 2)(k - 3)}{4!}$$

$a_0 = 0$, $a_1 = 1$, $a_2 = 2^4 - 2(1) = 14$,

$a_3 = 3^4 - 3(2^4) + 3(1) = 36$,

$a_4 = 4!$. Whence,

$$\sum_{k=1}^{n} k^4 = \frac{(n + 1)(n)}{2} + \frac{7(n + 1)(n)(n - 1)}{3} + \frac{6(n + 1)(n)(n - 1)(n - 2)}{4} + \frac{(n + 1)(n)(n - 1)(n - 2)(n - 3)}{5} .$$

6. (a) Establish Equation (ii) of Number 4.

Since $a_0 , a_1 , \dots , a_r$ are defined by the equation

$$P(k) = a_0\binom{k}{0} + a_1\binom{k}{1} + \dots + a_r\binom{k}{r} ,$$

we have the following r linear equations for the a_1's :

$$P(0) = a_0\binom{0}{0} ,$$

$$P(1) = a_0\binom{1}{0} + a_1\binom{1}{1} ,$$

$$P(2) = a_0\binom{2}{0} + a_1\binom{2}{1} + a_2\binom{2}{2} ,$$

$$\vdots \qquad\qquad \vdots$$

$$P(r) = a_0\binom{r}{0} + a_1\binom{r}{1} + \dots + a_r\binom{r}{r} .$$

Our proof is by mathematical induction. Assume that

$$\text{(A)}\quad a_n = P(n)\binom{n}{0} - P(n-1)\binom{n}{1} + \ldots + (-1)^n P(0)\binom{n}{n} = \sum_{k=0}^{n} (-1)^k P(n-k)\binom{n}{k}$$

is valid for $n = 0, 1, 2, \ldots, m-1$. We now wish to show that the expression for a_n is also valid for $n = m$. This is equivalent to showing

$$\text{(B)}\quad P(m) = a_0\binom{m}{0} + a_1\binom{m}{1} + \ldots + a_m\binom{m}{m} = \sum_{j=0}^{m} a_j\binom{m}{j}$$

(for the values of a_n given above, $n = 1, 2, \ldots, m$. This will involve manipulations on double series.

$$\sum_{j=0}^{m} a_j\binom{m}{j} = \sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}(-1)^k P(j-k)\binom{j}{k}$$

(by substituting for a_j in (A)).

Now let $k = j - i$. Then,

$$\sum_{j=0}^{m} a_j\binom{m}{j} = \sum_{j=0}^{m}\sum_{i=0}^{j}\binom{m}{j}(-1)^{j-i}P(i)\binom{j}{j-i}\ .$$

Noting that $\binom{j}{j-i} = \binom{j}{i}$ and interchanging order of summation, we get

$$\sum_{j=0}^{m} a_j\binom{m}{j} = \sum_{i=0}^{m}\sum_{j=i}^{m}\binom{m}{j}(-1)^{j-i}P(i)\binom{j}{i}$$

$$= \sum_{i=0}^{m} P(i) \sum_{j=i}^{m} (-1)^{j-i}\binom{m}{j}\binom{j}{i}\ .$$

Since $\binom{m}{j}\binom{j}{i} = \binom{m}{i}\binom{m-i}{j-i}$,

$$\sum_{j=i}^{m} (-1)^{j-i}\binom{m}{j}\binom{j}{i} = \sum_{j=i}^{m} (-1)^{j-i}\binom{m}{i}\binom{m-i}{j-i}\ .$$

Now let $j = i + r$, which reduces the last summation to

$$\binom{m}{i}\sum_{r=0}^{m-r} (-1)^r\binom{m-i}{r} = 0 \text{ if } i \neq m$$

$$= 1 \text{ if } i = m$$

(see Exercises A3-2a, No. 15b).

Finally, $\sum_{j=0}^{m} a_j\binom{m}{j} = P(m)$ which was to be shown.

Since our inductive hypothesis (A) is valid for $n = 0$, it is valid for all n.

(b) Show that a_m is zero for $m > r$.

Suppose we wanted the equation $F(x) = a_0\binom{x}{0} + a_1\binom{x}{0} + \dots + a_m\binom{x}{m}$ (where m is any number $> r$), to be satisfied for $x = 0, 1, 2, \dots, m$ where $F(x)$ is some given function. By setting $x = 0, 1, 2, \dots, m$ in turn the a_1's will have to satisfy

$$\begin{aligned} F(0) &= a_0\binom{0}{0}, \\ F(1) &= a_0\binom{1}{0} + a_1\binom{1}{1}, \\ &\vdots \\ F(m) &= a_0\binom{m}{0} + a_1\binom{m}{1} + \dots + a_m\binom{m}{m}. \end{aligned}$$

It follows (from algebra) that this system of $(m + 1)$ linear equations in $(m + 1)$ unknowns has a unique solution for all $F(x)$. By our inductive argument in part (a), the solution is given as

$$a_n = F(n)\binom{n}{0} - F(n-1)\binom{n}{1} + \dots + (-1)^n F(0)\binom{n}{n}$$

for $n = 0, 1, 2, \dots, m$.

If we now choose $F(x)$ to be the polynomial $P(x)$ of degree r in part (a), then $P(x)$ is identical to

$$a_0\binom{x}{0} + a_1\binom{x}{1} + \dots + a_r\binom{x}{r}$$

(from Problem 4). It then follows that

$$a_{r+1}\binom{x}{r+1} + a_{r+2}\binom{x}{r+2} + \dots + a_m\binom{x}{m}$$

vanishes for $x = 0, 1, 2, \dots, m$. If a polynomial of degree m vanishes for $m + 1$ different values it must identically vanish. Therefore,

$$a_{r+1} = a_{r+2} = \dots = a_m = 0$$

for all $m > r$.

Teacher's Commentary

Appendix 4

FUNCTIONS CONTINUOUS ON AN INTERVAL

Solutions Exercises A4-1

1. Is the continuity of f essential to the hypothesis of the Boundedness Theorem?

   Yes. Consider

$$f : x \longrightarrow f(x) = \begin{cases} \frac{1}{x} , & x \neq 0 , \\ 0 , & x = 0 , \end{cases} \quad \text{on } [0,1] .$$

2. Can a discontinuous function whose domain is a closed interval be bounded?

   Yes. Consider $x \longrightarrow [2x]$ on $[0,1]$, or $x \longrightarrow [nx]$.

3. Do Numbers 1 and 2 amount to the same question?

   No. Number 1 asks, Is the Boundedness Theorem true if we drop the hypothesis of continuity? while Number 2 asks, Is continuity necessary for boundedness?

4. Can a nonconstant function whose domain is the set of real numbers be bounded?

   Yes; e.g., $x \longrightarrow \sin x$ or $x \longrightarrow \dfrac{x^2}{x^2+1}$.

5. Give an example of a monotone function on $[0,1]$ with exactly n points of discontinuity.

   $x \longrightarrow [nx]$.

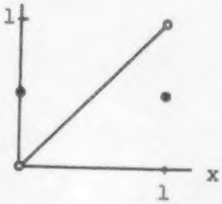
∧ 6. Can a monotone function on $[0,1]$ have infinitely many points of discontinuity? Justify your answer.

   Yes; e.g.,

$$x \longrightarrow \begin{cases} \dfrac{1}{\left[\frac{1}{x}\right]} , & x \neq 0 \\ 0 , & x = 0 . \end{cases}$$

7. (a) Give an example of a bounded function f defined on $[0,1]$ such that f has no extreme values.

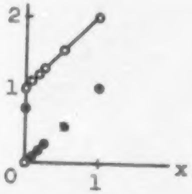
$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 , \\ \frac{1}{2} & \text{for } x = 0 , 1 . \end{cases}$$



(b) Repeat (a) with the extra condition that f have an inverse.

$$f(x) = \begin{cases} x + 1 & \text{for } x \neq \frac{1}{n} , \quad n \text{ a positive integer,} \\ x & \text{for } x = \frac{1}{n} , \\ \frac{3}{4} & \text{for } x = 0 . \end{cases}$$

(Domain: $0 \leq x \leq 1$.)



8. Give an example of a function f defined in the interval $[0,1]$ such that:

(a) f has neither an upper or lower bound.

$$f_1(x) = \begin{cases} \tan(x + \frac{1}{2})\pi & \text{for } 0 < x < 1 , \\ 0 & \text{for } x = 0 , 1 . \end{cases}$$

$$f_2(x) = \begin{cases} \frac{1}{x} + \frac{1}{x - 1} & \text{for } 0 < x < 1 , \\ 0 & \text{for } x = 0 , 1 . \end{cases}$$

(b) f has a lower bound but no upper bound.

$$f_1(x) = \begin{cases} \frac{1}{x} & \text{for } 0 < x \leq 1 , \\ 0 & \text{for } x = 0 . \end{cases}$$

$$f_2(x) = \begin{cases} \tan(x + \frac{\pi}{2}) & \text{for } 0 < x \leq \frac{1}{2} , \\ 0 & \text{for } \frac{1}{2} \leq x \leq 1 . \end{cases}$$

(c) f achieves the upper and lower bounds an infinite number of times. (For this case give a function that is not constant in any interval.)

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{for } 0 < x \leq 1 , \\ 0 & \text{for } x = 0 . \end{cases}$$

9. Show that any function f satisfying Number 8(c) cannot be continuous in the entire interval.

Successively divide the interval into halves, and in each case choose one half interval in which the function assumes its upper and lower bounds (M and m) infinitely often. By the Nested Interval Property we obtain a number s with the property that in every neighborhood of s , f assumes the values M and m . But this is impossible at a point of continuity unless $M = m$.

10. Give an example of a function f defined on $[0,1]$ such that f takes on every value between 0 and 1 once and only once but is discontinuous for all x .

$$f_1(x) = \begin{cases} x & \text{for } x \text{ irrational} \\ 1 - \frac{p}{q} & \text{for } x = \frac{p}{q} \text{, rational,} \end{cases}$$

is discontinuous at all points but $x = \frac{1}{2}$. To rid ourselves of this point of continuity we only need to interchange two values; e.g., set $f(0) = \frac{1}{2}$ and $f(\frac{1}{2}) = 1$. So

$$f_2(x) = \begin{cases} x & \text{for } x \text{ irrational ,} \\ 1 - \frac{p}{q} & \text{for } x = \frac{p}{q} \text{ rational, } \neq 0 , \frac{1}{2} , \\ \frac{1}{2} & \text{for } x = 0 , \\ 1 & \text{for } x = \frac{1}{2} , \end{cases}$$

suffices.

11. Show that a function which is increasing in some neighborhood of each point of an interval (a,b) is an increasing function in (a,b). (Note: The text problem using the concept "increasing at a point" is out of place. It is a problem given in Exercises A7-2.)

Consider the set A of points α in (a,b) such that f is increasing in (a,α) . Call $\bar{\alpha}$ the least upper bound of A . Then for $\beta > \alpha$, f is not increasing in (a,β) .

We are given that f is increasing in a neighborhood of $\bar{\alpha}$ if $\bar{\alpha} < b$. Therefore, for some h , f is increasing in $(\bar{\alpha} - h, \bar{\alpha} + h)$. Hence, f is increasing in $(a , \bar{\alpha} + h)$. But this means that $\bar{\alpha} + h$ is in A , while α is an upper bound of A . So $\bar{\alpha} \geq b$ and f is increasing in (a,b) .

12. A function ϕ is said to be weakly increasing "on the right" in a neighborhood of a if $\phi(x) \geq \phi(a)$ for all x in the neighborhood satisfying $x \geq a$.

(a) Show that if ϕ is continuous and weakly increasing on the right of all points in (b,c) , then ϕ is weakly increasing in (b,c) .

If ϕ is not weakly increasing on (a,b) , there exist points x_1 , $x_2 \in (a,b)$ such that $x_1 < x_2$ and $\phi(x_1) > \phi(x_2)$. Since ϕ is continuous, it has a maximum η on $[x_1, x_2]$. Let $\xi = \sup\{x : \phi(x) = \eta ,\ x \in [x_1,x_2]\}$. By continuity, $\phi(\xi) = \eta$. Now $\phi(x_2)$ is not the maximum since $\phi(x_2) < \phi(x_1)$; consequently, $x_2 \neq \xi$. Since $\phi(x) \leq \phi(\xi)$ for $x > \xi$ in some neighborhood of ξ , the condition that ϕ is everywhere increasing on the right is contradicted at $x = \xi$.

(b) Show by a counter-example that (a) does not necessarily hold if ϕ is discontinuous.

$$f(x) = \begin{cases} x & \text{, for } x \in [0,1] , \\ x - 1 & \text{, for } x \in [1,2] . \end{cases}$$

13. A function has the property that for each point of an interval where it is defined, there is a neighborhood in which the function is bounded. Show that the function is bounded over the whole interval. (This is an example where a local property implies a global one. It is clear that the global property here implies the local one.)

Let I be the interval for which f is locally bounded and let a and b be the respective left and right endpoints of I (no implication that I is either open or closed). Let A be the set of points consisting of the point a and those points α of I for which $f(x)$ is bounded on the interval $I_\alpha = I \cap \{x : x \leq \alpha\}$. Take $\bar{\alpha} = \sup A$. If $\alpha > \bar{\alpha}$, then f cannot be bounded on I_α . If $\bar{\alpha} = b$, then f is bounded on I . If $\bar{\alpha} < b$, then f is bounded on a neighborhood of $\bar{\alpha}$. It follows that f is bounded on the union of A and this neighborhood, contradicting that there is no interval I_α with $\alpha > \bar{\alpha}$ for which f is bounded.

14. Give an example of a function defined everywhere in a closed interval but unbounded in the neighborhood of every point of the interval. (Suggestion: See Exercises 3-5, No. 15.)

$$f(x) = \begin{cases} 0 \text{ , if } x \text{ is irrational,} \\ q \text{ , if } x \text{ is rational and } x = \frac{p}{q} \text{ } (\frac{p}{q} \text{ in lowest terms.} \end{cases}$$

Solutions Exercises A4-2

1. Can a discontinuous function have the Intermediate Value Property? Give examples.

Yes; e.g., $x \longrightarrow x - [x]$.

Λ 2. Let the function f be the derivative of a function g . Prove that f has the Intermediate Value Property.

Let $r_h(x) = \frac{g(x + h) - g(x)}{h}$. Let $f(a) = \alpha$ and $f(b) = \beta$. Take γ between α and β . Let $\epsilon = \frac{1}{2} \min(|\gamma - \alpha| , |\gamma - \beta|)$. Then we can find δ_1 , δ_2 so that $|r_h(a) - \alpha| < \epsilon$ if $0 < h < \delta_1$, and $|r_h(b - h) - \beta| < \epsilon$ if $0 < h < \delta_2$. Take h less than both δ_1 and δ_2. Then γ is between $r_h(a)$ and $r_h(b)$. But $r_h(x)$ is continuous and hence has the Intermediate Value Property. Thus for some x between a and b , $r_h(x) = \gamma$. But then, by the Law of the Mean, there is a point ξ between x and $x + h$ for which $f(\xi) = g'(\xi) = \frac{g(x + h) - g(x)}{h} = \gamma$.

Λ 3. Given the half-circle $y = \sqrt{1 - x^2}$, it can be shown that chords parallel to the x-axis of length $\frac{1}{n}$ exist where n is any positive integer.

This result can be generalized to any continuous function taking on the value 0 at 0 and 1 . Chords which intersect the curve, or lie entirely outside the curve, or coincide with the curve are permitted. Prove this.

The problem may be stated analytically as follows. Prove that if f is continuous on $[0,1]$ and $f(0) = f(1) = 0$, then for each positive integer n we can find an x in $[0,1]$ such that $f(x) = f(x + \frac{1}{n})$. The function $g(x) = f(x + \frac{1}{n}) - f(x)$ defined on $[0 , 1 - \frac{1}{n}]$ is continuous. It is impossible that $g(x) > 0$ for x in $[0 , 1 - \frac{1}{n}]$, since if it were, then $f(1) = g(\frac{0}{n}) + g(\frac{1}{n}) + \dots + g(1 - \frac{1}{n}) > 0$ which contradicts $f(1) = 0$.

Similarly, we cannot have $g(x) < 0$ for x in $[0, 1 - \frac{1}{n}]$. Hence g changes sign, and by the Intermediate Value Theorem there is an x in $[0, 1 - \frac{1}{n}]$ for which $g(x) = 0$.

Solutions Exercises A4-3

1. Show that the Weierstrass function is not monotone in any interval.

Given any interval I, we can choose n large enough so that we can find two points $\frac{j}{3^n}$ and $\frac{j+1}{3^n}$ in I. Then by the construction of f, if $f(\frac{j}{3^n}) < f(\frac{j+1}{3^n})$, then $f(\frac{j}{3^n} + \frac{1}{3^{n+1}}) > f(\frac{j}{3^n} + \frac{2}{3^{n+1}})$, and if $f(\frac{j}{3^n}) > f(\frac{j+1}{3^n})$, then $f(\frac{j}{3^n} + \frac{1}{3^{n+1}}) < f(\frac{j}{3^n} + \frac{2}{3^{n+1}})$. Hence f is not monotone on I.

Teacher's Commentary

Appendix 5

IMPLICITLY DEFINED FUNCTIONS AND THEIR DERIVATIVES

When we speak about a function, F, of two variables we would be more consistent if we wrote

$$F : (x,y) \longrightarrow F\big((x,y)\big) .$$

Having said this, we can afford the convenience of the sloppier notation

$$F : (x,y) \longrightarrow F(x,y) .$$

Since a linear δ-neighborhood of a consists of all points x whose distance to a is less than δ, it might seem more natural to define a planar δ-neighborhood of (a,b) to consist of all (x,y) such that

$$\sqrt{(x-a)^2 + (y-b)^2} < \delta .$$

Since every square neighborhood contains a circular neighborhood and vice versa, the two notions are equivalent. For our purposes, it has been more convenient to work with square neighborhoods.

Solutions Exercises A5

1. Find conditions for which the equations

$$a_1t + b_1x + c_1y + d_1 = 0$$

$$a_2t + b_2x + c_2y + d_2 = 0$$

determine x and y as functions of t.

From the two equations we easily obtain

(i) $(a_2c_1 - a_1c_2)t + (b_2c_1 - b_1c_2)y + b_2d_1 - b_1d_2 = 0$

and

(ii) $(a_2c_1 - a_1c_2)t + (b_2c_1 - b_1c_2)x + c_1d_2 - c_2d_1 = 0$.

Hence, if $b_2c_1 - b_1c_2 \neq 0$, we find

$$y = -\frac{(a_1b_2 - a_2b_1)t + b_2d_1 - b_1d_2}{b_2c_1 - b_1c_2}$$

$$x = -\frac{(a_2c_1 - a_1c_2)t + c_1d_2 - c_2d_1}{b_2c_1 - b_1c_2} \quad .$$

2. Consider linear approximations to F and G, then use Number 1 to generalize Theorem A5b to the situation $F(t,x,y) = 0 = G(t,x,y)$.

Suppose $F(t_0,x_0,y_0) = 0 = G(t_0,x_0,y_0)$, and suppose that F and G have representations

$$\begin{aligned} F(t,x,y) &= a_1(t - t_0) + b_1(x - x_0) + c_1(y - y_0) \\ &\quad + (t - t_0)\epsilon_1(t,x,y) + (x - x_0)\epsilon_2(t,x,y) \\ &\quad + (y - y_0)\epsilon_3(t,x,y) \ , \end{aligned}$$

$$\begin{aligned} G(t,x,y) &= a_2(t - t_0) + b_2(x - x_0) + c_2(y - y_0) \\ &\quad + (t - t_0)\epsilon_4(t,x,y) + (x - x_0)\epsilon_5(t,x,y) \\ &\quad + (y - y_0)\epsilon_6(t,x,y) \ , \end{aligned}$$

where

$$\lim_{\substack{t\to t_0 \\ x\to x_0 \\ y\to y_0}} \epsilon_i(t,x,y) = 0 \ , \ i = 1 \ , 2 \ , \ldots \ , 6 \ .$$

Then if $b_2c_1 - b_1c_2 \neq 0$, unique functions $t \longrightarrow f(t)$, $t \longrightarrow g(t)$ are determined such that

(i) $$F\big(t,f(t),g(t)\big) = G\big(t,f(t),g(t)\big) = 0 \ ,$$

(ii) $$f'(t_0) = -\frac{a_2c_1 - a_1c_2}{b_2c_1 - b_1c_2} \ ,$$

(iii) $$g'(t_0) = -\frac{a_1b_2 - a_2b_1}{b_2c_1 - b_1c_2} \ .$$

This method can be generalized to

(1) $$F_i(x_1,x_2, \ldots, x_n \ ; y_1,y_2, \ldots, y_m) = 0 \ , \ i = 1,2, \ldots, m \ ,$$

where, under appropriate conditions, (1) uniquely determines $y_1 , \ldots , y_m$ as functions of $x_1 , \ldots , x_n$. The theorem describing this situation is called the Implicit Function Theorem and is discussed in advanced texts.

3. **Prove that the following equations uniquely define y as an implicit function of x near the points indicated. (Note: The wording of the problem has been slightly changed from that in the text.)**

(a) $3x^2 + xy + 3y^2 = 7$, $(\sqrt{\frac{7}{3}}, 0)$.

$$F(x,y) = 3x^2 + xy + 3y^2 - 7 = 0$$

$$\begin{aligned} \phi_\xi(y) &= 3\xi^2 + \xi y + 3y^2 - 7 \\ &= 3(y^2 + \frac{3y}{3} + \frac{\xi^2}{36}) + C \\ &= 3(y + \frac{\xi}{6})^2 + C \ . \end{aligned}$$

$\phi_\xi(x)$ for all ξ are polynomials, hence continuous, and are monotone in the neighborhood of all points (x,y) on $F(x,y) = 0$ such that $y \neq -\frac{\xi}{6}$; for $\xi = \sqrt{\frac{7}{3}}$, $\phi_\xi(y)$ is monotone in a neighborhood of $y = 0$. All $\psi_\eta(x)$ are polynomials in x , hence continuous. So $F(x,y)$ at $(\sqrt{\frac{7}{3}}, 0)$ meets the conditions of the Implicit Function Theorem.

(b) $F(x,y) = x^3 - 3xy + y^3 = 0$.

$\phi_\xi(y)$ and $\psi_\eta(x)$ are polynomials in x and y , respectively, for all ξ , η , hence continuous over their entire domain.

$$\begin{aligned} \phi_\xi(y) &= y^3 - 3\xi y + \xi^3 \ , \\ \phi_\xi'(y) &= 3y^2 - 3\xi \ , \\ \phi_\xi'(y) > 0 &\text{ for } y^2 > \xi \ , \\ \phi_\xi'(y) < 0 &\text{ for } y^2 < \xi \ . \end{aligned}$$

Since ϕ_ξ' is continuous, $\phi_\xi'(y) > 0$ or $\phi_\xi'(y) < 0$ in some neighborhood of all points such that $y^2 \neq \xi$. Therefore, ϕ_ξ is monotone in a neighborhood of each of these points. Since $(\frac{3}{2})^2 \neq \frac{3}{2}$, we are finished.

(c) $F(x,y) = x \cos xy = 0$.

$\phi_\xi(y) = \xi \cos \xi y$, a continuous function of y for all ξ .
$\phi_\xi'(y) = -\xi^2 \sin \xi y$. For ξ close to 1 , y close to $\frac{\pi}{2}$, $\sin \xi y > 0$ and $\phi_\xi'(y) < 0$. Then $\phi_\xi'(y)$ is decreasing in some

neighborhood of $(1, \frac{\pi}{2})$.

$\psi_\eta(x) = x \cos \eta x$ is a continuous function of x for all x , η . Then we have shown that the conditions of Theorem A5a are met at $(1, \frac{\pi}{2})$.

4. Find the first and second derivatives of the implicit functions defined by (a), (b), and (c) of Number 3. (Note: The wording is slightly different from that in the text.)

(a) $3x^2 + xy + 3y^2 = 7$

(i) $6x + xy' + y + 6yy' = 0 \qquad y' = -\frac{6x + y}{x + 6y}$;

(ii) $6 + xy'' + y' = y' + 6yy'' + 6y'y) = 0$

$$y'' = -\frac{6 + 2y' + 6(y')^2}{x + 6y} = \frac{210x^2 + 70xy + 210y^2}{(x + 6y)^3} .$$

(b) $x^3 - 3xy + y^3 = 0$.

(i) $3x^2 - 3xy' - 3y + 3y^2y' = 0 . \quad y' = \frac{x^2 - y}{x - y^2}$,

(ii) $2x = xy'' - y' = y' + y^2y'' + 2y(y')^2 = 0$

$$y'' = \frac{2x - 2y' + 2y(y')^2}{x - y^2} = \frac{2x^4y - 6x^2y^2 + 2xy^4 + 2xy}{(x - y^2)^3}$$

(c) $x \cos xy = 0$.

We note that the given equation implies $x = 0$ or $\cos xy = 0$, and $\cos xy = 0 \Longrightarrow xy = \frac{\pi}{2} + n\pi$, n an integer. Hence

(i) $xy' + y = 0 \Longrightarrow y' = -\frac{y}{x}$,

(ii) $xy'' + y' + y' = 0 \Longrightarrow y'' = -\frac{2y'}{x} = \frac{2y}{x^2}$.

5. Show that there is no unique solution for y in Number 3(b) in any neighborhood of the point $(2^{2/3}, 2^{1/3})$. (Note: "in any neighborhood of" replaces "near" used in the text.)

Using the same analysis as in 3(b), we can determine a unique function $y \to x$ satisfying the equation near $(2^{2/3}, 2^{1/3})$. Moreover, $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$ is positive to the right of $2^{1/3}$ and negative to the left. Hence the function has a maximum at $2^{1/3}$. Thus in any neighborhood of $(2^{2/3}, 2^{1/3})$, if $x < 2^{2/3}$ there are two values of y for every value of x.

6. For possible inverse functions of $f : x \longrightarrow y$, show that

$$\frac{d^2x}{dy^2} = -\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3} .$$

Differentiating $y = f(x)$ implicitly with respect to y, we have $1 = f'(x)\frac{dx}{dy}$. Differentiating again we obtain

$$0 = f'(x)\frac{d^2x}{dy^2} + \frac{dx}{dy} f''(x) \frac{dx}{dy} .$$

Hence,

$$\frac{d^2x}{dy^2} = -\frac{\left(\frac{dx}{dy}\right)^2 f''(x)}{f'(x)}$$

$$= -\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3} ,$$

since $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$ and $f'(x) = \frac{dy}{dx}$.